Using long-run consumption-return correlations to test asset pricing models

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A R T I C L E   I N F O

Article history:
Received 4 December 2010
Revised 10 April 2012
Available online 23 April 2012

JEL classification:
G10
G12

Keywords:
Long-run risk
Habit
Forward-looking

A B S T R A C T

This paper examines a new set of implications for existing asset pricing models regarding the correlation between returns and consumption growth over both the short run and the long run. The findings suggest that external habit formation models face a challenge in producing two robust facts in aggregate data, namely, that stock market returns lead consumption growth, and that the correlation between returns and consumption growth is higher at low frequencies. To reconcile these facts with a consumption-based model, I demonstrate the need for focusing on models that contain a forward-looking consumption component, i.e., models that allow for both trend and cyclical fluctuations in consumption, and that link returns to cyclical fluctuations in consumption. Long-run risk models provide examples of models that contain this consumption component.

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1. Introduction

It is well known that both the habit model of Campbell and Cochrane (1999) (CC hereafter) and the long-run risk model of Bansal and Yaron (2004) (BY hereafter) can account for many asset pricing phenomena simultaneously. However, these two classes of models operate through different mechanisms and constitute fundamentally different views of the sources and the pricing of risk. In this paper, I propose a simple and intuitive way to distinguish between the mechanisms underlying the sources of risk in various consumption-based asset pricing models.

In this paper, I first examine the joint dynamics of consumption and market excess returns in the data and document that excess stock market returns positively lead consumption growth. I also show that long-run consumption-return correlations are much higher than those in the short run. Then, I develop tests by comparing the model-implied joint time-series properties of consumption and returns with those documented in the data. Moreover, I highlight the key ingredients in asset pricing models that are necessary to reproduce the observed joint time-series properties of consumption and asset returns in the data.

Specifically, I show that reasonably calibrated versions of the CC model imply that consumption leads asset returns and that long-run correlations are lower than short-run correlations. The intuition for this finding is simple. In external habit models, the presence of a habit level that depends on past consumption growth implies that past consumption determines...
the effective risk aversion of the representative agent. Thus, negative past consumption shocks should predict high future excess returns, because of the negative dependence of risk aversion on past consumption innovations. Moreover, given a long enough horizon, this negative covariation between expected excess returns and past consumption growth will drive correlations down, potentially even below zero. Interestingly, this remains the case even when I allow consumption to have predictable components inside a CC model. This is because the driving force in the model is still the surplus ratio. Expected returns are mostly driven by this variable, which is the weighted average of past consumption growth. Hence, consumption still leads returns in external habit models even in the presence of predictable consumption growth. Therefore, external habit models have difficulty matching the joint consumption-return dynamics.

By contrast, I show that for models that feature both a stochastic trend and a cycle in consumption, and that link asset returns to the cyclical consumption component, it is easier to match the observed joint consumption-return dynamics. The long-run risks model provides such an example. Intuitively, in such a model, a high return is observed because a higher growth rate of consumption is anticipated. Thus, asset returns lead consumption. In addition, this positive association between asset returns and expected consumption growth results in a pattern of increasing long-run correlations between consumption and returns as the horizon increases. Accordingly, I conclude that models that derive variations in returns from expectations of future consumption growth are better suited to match this pattern in the data.

To sharpen the results even further, in addition to performing the analysis in the time domain, I also consider the joint time-series properties of consumption and returns in the frequency domain. Besides allowing me to examine a richer set of time-series implications, the frequency domain facilitates the derivation of analytical results for the models under consideration. For example, this approach allows me to show more clearly how the presence of a persistent habit process leads to an attenuated correlation between consumption and returns at low frequencies. Specifically, through a log-linearization argument, I show that as long as the external habit model produces a countercyclical risk premium or a procyclical price–dividend ratio, the model implies that the covariation between consumption and returns is greater in high-frequency components, whereas in the data, the opposite occurs.

This paper adds to the literature that analyzes the properties of the habit model and the long-run risk model along other important dimensions. For example, Otrok et al. (2002) argue that habit agents are much more averse to high-frequency fluctuations than they are to low-frequency fluctuations, and the size of the equity premium is determined by a relatively insignificant amount of high-frequency volatility in U.S. consumption. Santos and Veronesi (2006) show that the external habit model generates counterfactual predictions in the cross section of stock returns. Lustig et al. (2007) find that the external habit model and long-run risk model have different implications for the wealth–consumption ratio. In contemporaneous work, Bansal et al. (2009b) explore the distinction between habit models and long-run risk models by looking at the predictability of the price–dividend ratio by past consumption, and they also highlight the importance of forward-looking behavior in a model.

In addition, Backus et al. (2010) show that asset returns lead consumption, and they highlight the key ingredients needed to reproduce these patterns. This paper complements Backus et al. (2010) by examining the inability of external habit models to match the joint consumption-return dynamics and by investigating long-run correlations in addition to the lead-lag relation. Finally, my study is also related to the literature that brings additional important aspects of the data for models to match. Alvarez and Jermann (2005), for example, take into account of return properties of the long-term zero-coupon bonds and derive restrictions on the permanent and transitory components of general pricing kernels. Backus et al. (2011) explore a similar problem by using a convenient entropy analysis. Bakshi and Chabi-Yo (forthcoming) extend the analysis of Alvarez and Jermann (2005) on the bounds of pricing kernels by considering a larger asset space.

The remainder of the paper is organized as follows. In Section 2, I document the lead-lag relation and the long-run correlation between consumption and asset returns in the data. In Section 3, I present two alternative asset pricing models and analytically explore the model-implied joint dynamics of consumption and asset returns. Section 4 provides a quantitative assessment for the three models. I then highlight in Section 5 the important ingredients necessary to reproduce the patterns observed in the data. Finally, Section 6 concludes the paper. All the proofs are provided in the Internet Appendix.

2. Joint dynamics of consumption and returns

In this section, I document a few stylized facts on the relation between consumption and asset returns. Quarterly real seasonally adjusted consumption per capita from 1947Q1 to 2009Q4 is obtained from the U.S. Bureau of Economic Analysis (BEA). Consumption is for nondurables and services excluding shoes and clothing. The corresponding quarterly excess value-weighted market return is taken from the CRSP VW index.

2.1. Granger causality test

I conduct a Granger causality test to examine the lead-lag relation between consumption and asset returns. To implement this test, I assume an autoregressive lag length of two and estimate the following equation by ordinary least squares (OLS):

\[ r_t = c_1 + \alpha_1 r_{t-1} + \alpha_2 r_{t-2} + \beta_1 g_{c,t-1} + \beta_2 g_{c,t-2} + u_{r,t}, \]

where \( r_t \) is the quarterly market excess return and \( g_{c,t} \) is the quarterly consumption growth rate. I then conduct an F test of the following null hypothesis: \( H_0: \beta_1 = \beta_2 = 0 \). Similarly, I can estimate the OLS,
and then conduct an F test of the null hypothesis $H_0$: $\eta_1 = \eta_2 = 0$. The $p$-value of the Granger causality test of consumption Granger-causing returns is 0.17, whereas the $p$-value of the Granger causality test of returns Granger-causing consumption is $6.17 \times 10^{-4}$. Hence, the statistical test rejects the null that stock market returns do not Granger-cause consumption, whereas the test cannot reject the null that consumption does not Granger-cause stock market returns. For the annual data, the results are even stronger. Thus, the evidence suggests that asset returns lead consumption. In addition, both coefficients $\eta_1$ and $\eta_2$ are positive and statistically significant with $t$-statistics of 2.30 and 3.02, respectively. Thus, asset returns positively predict future consumption growth.

In untabulated analysis, I also perform Granger causality tests for the Fama-French 25 portfolios and the consumption growth rate. The portfolio-level analysis is largely consistent with the aggregate analysis. That is, the 25 portfolio returns lead consumption growth (all the $p$-values are less than 0.05), but consumption growth does not lead any of the 25 portfolio returns. These results are also consistent with the findings in Backus et al. (2010). Using cross-correlation analysis, they find that equity returns sorted by size and book-to-market lead industrial production.

2.2. Cumulative correlations

I calculate the correlation between consumption and asset returns over different horizons. The correlation between cumulative consumption growth and cumulative excess market returns, $corr(\sum_{j=1}^{k} x_{t+j}, \sum_{j=1}^{k} r_{t+j})$, increases (almost monotonically) from 0.19 for $k = 1$ quarter to 0.59 for $k = 20$ quarters (see Fig. 2 for a graphic illustration). To further confirm the stronger correlation over the long run, I also perform a band-pass filtering analysis (e.g., Baxter and King, 1999). The band-pass filter is used to extract the low-frequency and high-frequency components of consumption and asset returns. The resulting correlations between the consumption growth rate and market returns at different frequencies are then calculated. For real data, the correlation is 0.17 for higher frequencies (with cycles between 2 and 12 quarters) and 0.42 for lower frequencies (with cycles longer than 12 quarters).

2.3. Spectral analysis

Since the focus of this paper is on the long-run (low-frequency) relation between consumption and returns, the most convenient way to proceed is to use bivariate spectral analysis. More important, spectral analysis allows a set of analytical results to be derived and allows me to analyze the correlations at different frequencies and the lead-lag relation between consumption and asset returns in a unified framework. Spectral analysis has been applied to asset pricing models by previous studies such as Daniel and Marshall (1999) and Otrok et al. (2002). Below, following Daniel and Marshall (1999), I give an intuitive explanation of the coherence, cospectrum, and phase spectrum that will prove useful for the rest of the analysis (see Brockwell and Davis, 1991, Chapter 11.7, for details).

The coherence of the consumption growth rate and stock market returns at frequency $\lambda$ measures the correlation between the cumulative consumption growth rate and returns at frequency $\lambda$. Essentially, coherence analysis splits each of the two series into a set of periodic components at different frequencies and then determines the correlation of a set of periodic components for the two series around each frequency. When the frequency is $\lambda$, the corresponding length of the cycle is $1/\lambda$ quarters. Hence, when $\lambda = 0.5$, the corresponding cycle is two quarters. Since coherence is always positive, the sign of the correlation at different frequencies cannot be determined from the coherence spectrum. To identify the sign of the correlation, the cospectrum needs to be examined. The cospectrum at frequency $\lambda$ can be interpreted as the portion of the covariance between consumption growth and asset returns that is attributable to cycles with frequency $\lambda$. Since the covariance can be positive or negative, the cospectrum can also be positive or negative. The slope of the phase spectrum at any frequency $\lambda$ is the group delay at frequency $\lambda$ and precisely measures the number of leads or lags between consumption growth and asset returns. When this slope is positive, consumption leads the market return. On the other hand, when this slope is negative, asset market returns lead consumption growth. Therefore, the coherence, cospectrum, and phase spectrum provide a convenient tool for analyzing the lead-lag relation and the correlations at different frequencies between two time series.

The left panels in Fig. 1 plot the cross spectra for the quarterly data, while the right panels present results for the annual data. The top left panel of Fig. 1 confirms Daniel and Marshall’s (1999) finding that the coherence between the quarterly consumption growth rate and quarterly excess market returns is much higher at low frequencies (around 0.08) than at high frequencies (around 0.5). Therefore, the comovement between consumption growth and asset market returns is much stronger at low frequencies. The middle left panel shows that most of the covariance comes from the low-frequency covariation. The cospectrum is normalized so that the area under the curve is one. This makes subsequent comparisons

\[ g_{c,t} = c_2 + y_1 g_{c,t-1} + y_2 g_{c,t-2} + \eta_1 r_{t-1} + \eta_2 r_{t-2} + u_{c,t}, \]
Fig. 1. Cross spectra for quarterly and annual data. The three left panels in this figure plot the nonparametric estimation of the coherence, normalized cospectrum, and phase spectrum between the quarterly consumption growth rate and quarterly excess stock market returns for the data. Quarterly real seasonally adjusted consumption per capita from 1947Q1 to 2009Q4 is obtained from the BEA. Consumption is for nondurables and services excluding shoes and clothing. The corresponding quarterly excess market return is taken from the CRSP VW index. The three right panels in this figure plot the nonparametric estimation of the coherence, normalized cospectrum, and phase spectrum between the annual consumption growth rate and annual excess stock market returns for the data. Annual real consumption per capita from 1930 to 2009 is obtained from the BEA. Consumption is for nondurables and services excluding shoes and clothing. The corresponding annual excess market return is taken from the CRSP VW index. The middle solid line is the estimation, and the other two dotted lines are the 95% confidence bands. The normalized cospectrum is calculated such that the area under the curve is one. The graph indicates that returns lead consumption, and comovements between consumption and returns are stronger at low frequencies.

among different models more transparent. The 95% confidence interval is also shown by the dotted line, and the confidence interval for the cospectrum is above zero at low frequencies. On the other hand, the high-frequency cospectrum is close to zero. The bottom left panel of Fig. 1 shows that the phase spectrum is monotonically decreasing at low frequencies. Since the middle left panel indicates that most of the covariation between consumption and returns comes from low frequencies, one needs only look at the slope of the phase spectrum at low frequencies to determine the lead-lag relation. Since this slope is negative at low frequencies, market returns lead consumption growth.\(^2\) This lead-lag relation confirms the previous Granger causality test.

Using annual data, the right three panels provide consistent results, with the same decreasing pattern and a comparable magnitude. Notice that in the right panels, a frequency \(\lambda = 0.3\) implies a cycle with length of 3.3 years. Hence, both the annual and quarterly data suggest that the covariation between consumption and returns is much stronger for cycles among different models more transparent. The 95% confidence interval is also shown by the dotted line, and the confidence interval for the cospectrum is above zero at low frequencies. On the other hand, the high-frequency cospectrum is close to zero. The bottom left panel of Fig. 1 shows that the phase spectrum is monotonically decreasing at low frequencies. Since the middle left panel indicates that most of the covariation between consumption and returns comes from low frequencies, one needs only look at the slope of the phase spectrum at low frequencies to determine the lead-lag relation. Since this slope is negative at low frequencies, market returns lead consumption growth.\(^2\) This lead-lag relation confirms the previous Granger causality test.

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\(^2\) Notice that the confidence interval for the phase spectrum is very wide for high frequencies. This is because the coherence between consumption and asset returns is very small for high frequencies. It is hard to determine which is leading when their comovement is very small.
longer than 3 years. The phase spectrum is monotonically decreasing for annual data, confirming that asset returns lead consumption growth.³

Some of the above empirical findings have also been documented by other studies. For example, Daniel and Marshall (1997, 1999) and Parker and Julliard (2005) show that long-run correlations between consumption and asset returns are stronger, and hence asset pricing models perform better at long horizons. In a contemporaneous study, Backus et al. (2010) show that excess returns on a broad range of assets lead a variety of business cycle activities, including consumption growth and industrial production growth. In sum, the evidence in this article and other studies strongly suggests that asset returns lead consumption and the correlation between consumption and asset returns is stronger over the long run. In this paper, I use these stylized facts to evaluate leading asset pricing models.

3. An analytical investigation

In this section, I briefly present two leading asset pricing models individually. I then quickly derive the analytical (approximate) model-implied relation between consumption growth and asset returns for each model. In the next section, I quantitatively compare the model implications side by side with numerical simulation. As will become clear later, most of the key results for these models crucially depend on the slow-moving state variables, and this slow-moving feature of the model has clear implications for the long run. Thus, exploring the low-frequency properties of these models is worthwhile.

3.1. The external habit formation model

I begin by setting up a standard external habit formation model following CC with an i.i.d. consumption growth rate. I also incorporate a cointegration constraint between dividends and consumption. Since the focus of this paper is on the long-horizon implications of different models, this cointegration constraint could potentially play an important role.⁴ The purpose of this cointegration constraint is to strengthen the long-run correlation between consumption and asset returns in the habit model. Thus, it increases the habit model’s ability to match the stronger long-run correlation between consumption and asset returns in the data.

3.1.1. Model setup

In this section, I set up an external habit persistence model that closely follows the specification of CC. Let \( c_t = \log(C_t) \) and \( d_t = \log(D_t) \) denote log real per capita values of consumption and the stock dividend. The consumption growth rate is generated according to

\[
g_{c,t} = \mu_c + \epsilon_{c,t},
\]

where \( \epsilon_{c,t} \) is i.i.d. normal with standard error \( \sigma_c \). The cointegrating constraint is that \( d_t - c_t \) is a stationary process, which evolves as follows:

\[
d_t = \mu_d + c_t + \delta_t,
\]

\[
\delta_t = \rho_\delta \delta_{t-1} + \epsilon_{\delta,t},
\]

where \( \epsilon_{\delta,t} \) is i.i.d. normal with standard deviation \( \sigma_\delta \) and \( \rho_\delta \) is the correlation between \( \epsilon_{c,t} \) and \( \epsilon_{\delta,t} \). This model assumes that \( 0 \leq \rho_\delta \leq 1 \). This setup of the dynamics of consumption and dividends is a direct extension of CC. Here, \( c_t \) and \( d_t \) are each \( I(1) \), and these two series are cointegrated except for when \( \rho_\delta = 1 \), in which case the model exactly reduces to that of CC. The agent is assumed to maximize the lifetime utility

\[
E_t \sum_{k=0}^{\infty} \beta^k \left( C_{t+k} - X_{t+k} \right)^{1-\gamma} - 1
\]

where \( C_t \) is real consumption, \( X_t \) is the agent’s habit level at time \( t \), \( \gamma \) is the risk aversion coefficient, and \( \beta \) is the time discount factor of the agent. The surplus ratio is defined as \( S_t = \frac{C_t - X_t}{c_t} \), and \( s_t = \log(S_t) \). The dynamics for \( s_t \) are given by

\[
s_{t+1} = (1 - \phi_s) \bar{s} + \phi_s s_t + \lambda(s_t) \epsilon_{c,t+1},
\]

where \( \bar{s} \) is the steady state of the log surplus ratio, and \( \phi_s \) determines the persistence of the surplus ratio (which also largely determines the persistence of the price–dividend ratio). Denote \( \bar{s} = \exp(\bar{s}) \). CC choose a special function form for \( \lambda(s_t) \) such that the risk-free rate is a constant. I use the same \( \lambda(s_t) \) as in CC.

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³ To reconfirm the results from the nonparametric estimation, a parametric method is also used to estimate the cross spectrum. In unreported results, I first estimate a VAR(2) for consumption growth and excess market returns. Then, by using the estimated parameter values, the cross spectrum between consumption growth and asset returns can be obtained analytically. The decreasing pattern in cross spectra remains in general.

⁴ A number of recent papers, including Bansal et al. (2005), Hansen et al. (2008), Bansal et al. (2009a), and Bekaert et al. (2009), suggest that dividends and consumption are stochastically cointegrated and that this cointegration is important for understanding asset pricing.
Before solving the CC model numerically, I first provide a log-linear approximation of the model, and I analytically derive some qualitative features of the model to gain better intuition for joint consumption-return dynamics. In the next section, I solve this model fully using the numerical approach of Wachter (2005) to assess its quantitative implications. The numerical solution confirms the intuition gained from the simple log-linearization argument below. In particular, under the log-linear approximation to the log price–dividend ratio (zt),

\[ z_t \approx a_0 + a_1 s_t + a_2 \delta_t, \]  

one can show that the approximated excess return on the dividend claim is given by

\[ r^{ex}_{t+1} \approx \alpha - \beta_S \sum_{j=1}^{\infty} \phi_j^{-1} s_{c,t+1-j} + \beta_c \epsilon_{c,t+1} + \beta_\delta \epsilon_{\delta,t+1}, \]  

where the expression for constants \( \alpha, \beta_S, \beta_c, \) and \( \beta_\delta \) are derived in the Internet Appendix with \( \beta_S > 0 \). Notice that \( \tilde{S}_t = \sum_{j=1}^{\infty} \phi_j^{-1} s_{c,t+1-j} \) is approximately the surplus ratio, \( S_t \).\(^5\) Thus, Eq. (6) implies that the expected excess return is mainly determined by the surplus ratio. This is intuitive, since the surplus ratio is the effective risk aversion of the representative agent. If \( \beta_S > 0 \), the equity premium is high when the surplus ratio is low. That is, the equity premium is countercyclical.

### 3.1.2. Implications for the joint dynamics of consumption and returns

Eq. (6) implies that the very mechanism that produces countercyclical risk premia in the model (\( \beta_S > 0 \)) is responsible for the negative correlation between future stock returns and lagged consumption growth rates. That is, consumption negatively leads asset returns in the external habit model, whereas asset returns positively lead consumption in the data. In addition, this very mechanism leads to decreasing long-run correlations between consumption and returns as the horizon increases. Note that Eq. (6) implies that future returns depend negatively on all the lagged consumption growth. As the horizon increases, this negative association plays a bigger role, and thus the long-run correlation decreases. As a consequence, while the data show an increasing pattern in long-run correlations, the external habit model predicts the exact opposite.

One can formalize the above intuition by using the frequency-domain analysis. In the Internet Appendix, I show the following proposition on the property of the cospectrum between consumption and excess returns in the habit formation model.

**Proposition 1.** Under the external habit formation model, the conditional equity premium is given by

\[ E_t(r^{ex}_{t+1}) \approx \alpha - \beta_S \sum_{j=1}^{\infty} \phi_j^{-1} s_{c,t+1-j}, \quad \text{and} \quad \beta_S \equiv \frac{a_1 (1 - \kappa_1 \phi_1)}{\tilde{S}}. \]  

where the constant \( \kappa_1 \in (0, 1) \). Furthermore, the cospectrum between consumption and excess returns, \( C_{sp}(\lambda) \), has the following property:

\[ \frac{dC_{sp}(\lambda)}{d\lambda} = \frac{\beta_S \sin(\lambda)}{2\pi (1 + \phi_1^2 - 2\phi_1 \cos(\lambda))^2 (1 - \phi_1^2)}. \]  

Thus, when \( \beta_S > 0 \), the cospectrum between consumption growth and asset returns is increasing in the frequency. That is, the portion of the covariance between consumption growth and asset returns that is attributable to cycles with frequency \( \lambda \) is increasing with the frequency \( \lambda \).

The above proposition implies that more covariations between consumption and excess returns are derived from components with long cycles, as long as the model produces countercyclical expected returns. Notice that the coefficient \( \beta_S \) is positive if and only if \( a_1 \) is positive (see Eq. (7)). From Eq. (5), a positive \( a_1 \) implies a procyclical price–dividend ratio. Thus, Proposition 1 implies that as long as the external habit persistence model produces a procyclical price–dividend ratio or a countercyclical risk premium, the cospectrum between consumption growth and asset returns is an increasing function of the frequency \( \lambda \). This result is at odds with the pattern in the data, as shown in the previous section.

Moreover, one can determine the sign of the correlations at very long horizons by examining the sign of the cospectrum between consumption growth and asset returns at the frequency \( \lambda = 0 \). In the Internet Appendix, I show that the cospectrum at frequency zero is given by

\[ C_{sp}(0) = \frac{\sigma_c^2}{2\pi} \left( 1 - a_1 \kappa_1 - \frac{a_1 (1 - \kappa_1)}{\tilde{S}} - \beta_\delta \frac{\sigma_{\delta \epsilon}}{\sigma_c^2} \right). \]  

Thus, the correlation at frequency \( \lambda = 0 \) (or the correlation at very long horizons) is negative if and only if

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\(^5\) Wachter (2006) uses a similar approximation for the surplus ratio by the past weighted average of consumption growth.
Thus, Eq. (11) holds as long as
\[ 1 - a_1 \kappa_1 - a_1 (1 - \kappa_1) + \beta_2 \frac{\sigma c_\delta}{\sigma c^2} < 0. \]  
\( \text{(10)} \)

When \( C_{sp}(0) < 0 \), the low-frequency correlations between the consumption growth rate and asset returns are negative (since the function \( C_{sp}(\lambda) \) is continuous in \( \lambda \)). This result is not supported by the data, as I have shown in the previous section. I formalize the above discussion by the following proposition.

**Proposition 2.** Under the log-linear approximation,
\[ z_t \approx a_0 + a_1 s_t + a_2 d_t, \]
and the following parameter restriction,
\[ 1 - a_1 \left( \kappa_1 + \frac{1 - \kappa_1}{S(1 - \phi_s)} \right) + \beta_2 \frac{\sigma c_\delta}{\sigma c^2} < 0, \]
\( \text{(11)} \)

there exists a frequency \( \lambda^* > 0 \) such that, for all \( \lambda < \lambda^* \), the correlation between the consumption growth rate and asset returns at frequency \( \lambda \) is negative.

It is important to examine when Eq. (11) is satisfied. First, notice that \( s_t = d_t - c_t \); hence, it is reasonable to assume a negative covariance between innovations in \( c_t \) and \( s_t \) (i.e., \( \sigma c_\delta < 0 \)). Thus, the fact that \( \beta_2 > 0 \) (shown in the Internet Appendix) implies that \( 1 - a_1 \left( \kappa_1 + \frac{1 - \kappa_1}{S(1 - \phi_s)} \right) < 0 \) is a sufficient condition for Eq. (11). Further notice that \( (\kappa_1 + \frac{1 - \kappa_1}{S(1 - \phi_s)}) > 0 \). Thus, Eq. (11) holds as long as \( a_1 \) is not too small.

Notice \( z_t \approx a_0 + a_1 s_t + a_2 d_t \) and thus \( a_1 \) measures the exposure of the log price–dividend ratio to the surplus ratio. If \( a_1 \) is too small, the price–dividend ratio is too smooth, and hence both the volatility of the excess return and the magnitude of the equity premium are too small in the model. Thus, for the model to make quantitative sense, \( a_1 \) cannot be too small, and the natural expectation is that the condition in Proposition 2 will be satisfied. Consequently, the habit model implies a negative correlation between consumption growth and asset returns at low frequencies. This result is at odds with the data, as I have shown in Section 2.

In sum, the external habit model is backward looking, since past consumption determines the current effective risk aversion and hence the expected future excess return. This exact backward-looking feature implies that consumption leads returns and the short-run covariation between consumption and returns are stronger than the long-run covariation. The opposite effect can be observed in the data.

### 3.2. Long-run risk models

In the previous subsection, I showed that external habit formation models have difficulty matching the relation between consumption and asset returns observed in the data. In this subsection, I consider another leading asset pricing model, the long-run risk model by Bansal and Yaron (2004). I consider two examples for the long-run risk model. The first one is the one-channel model in BY, which features predictable consumption growth without stochastic volatility. The second one is the two-channel BY model with both predictable consumption growth and stochastic consumption volatility. These examples help to demonstrate how long-run risk models can lead to the observed patterns in the correlation and lead-lag relation between consumption and asset returns.

**One-channel long-run risk model** In this example, I consider the one-channel long-run risk model, which features a simpler process for consumption and dividends. Thus, the algebra is transparent, and I can derive analytical implications for the joint dynamics of consumption and returns. Following the notation of BY, the consumption and dividend dynamics are given by
\[
\begin{align*}
g_{c.t+1} &= \mu + x_t + \sigma \eta_{t+1}, \\
x_{t+1} &= \rho x_t + \psi e \sigma e_{t+1}, \\
g_{d.t+1} &= \mu_d + \phi x_t + \psi_d \sigma u_{t+1},
\end{align*}
\]
\( \text{(12)} \)

where innovations \( \eta_t, e_t, \) and \( u_t \) are i.i.d. standard normal across time. BY assume that innovations \( \eta_t, e_t, \) and \( u_t \) are independent of each other. Below, I allow for a plausible small positive correlation between \( \eta_t \) and \( u_t \), denoted as \( \rho_{nu} \).

Following the same argument as in BY, one can show that, under Epstein and Zin (1989) recursive preference, excess returns on the dividend claim can be written approximately as
\[ r_{ex}^{c.t+1} \approx \bar{r} + \frac{\phi - \frac{1}{\psi}}{1 - \kappa_{1m} \rho} \kappa_{1m} \psi e \sigma e_{t+1} + \psi_d \sigma u_{t+1}, \]
\( \text{(13)} \)

where the constant \( \bar{r} \) is the expected risk premium, \( \psi \) is the intertemporal elasticity of substitution (IES), and \( \kappa_{1m} \in (0, 1) \) is the constant in the Campbell and Shiller (1988) log-linearization for returns on the dividend claim. Thus, in this one-channel model, excess return is an i.i.d. process. Taking Eqs. (12) and (13) together, it follows that
and the covariance is zero otherwise. Thus, for \( k > 1 \), \( \text{cov}(r_{t+1}^e, g_{t+k}) > 0 \) if and only if \( \frac{1}{\psi} > \phi > 0 \). BY choose \( \phi > 1 \) and IES \( \psi > 1 \). Thus, despite a constant risk premium in this one-channel model, excess returns can still positively predict future consumption growth, but consumption growth cannot predict future excess returns. This feature is consistent with the lead-lag relation in the historical data. Moreover, a closed-form solution for the cospectrum between consumption and asset returns can be derived in this case.

**Proposition 3.** The cospectrum between consumption and excess returns has the following property:

\[
\frac{d \text{Cov}(\lambda)}{d \lambda} < 0 \quad \text{for} \quad \lambda \in (0, \pi),
\]

if and only if the IES \( \psi > 1/\phi \). Moreover, under the assumption of zero correlations between innovations, as in the original BY model, the phase spectrum is always decreasing from low to high frequencies.

If I set \( \phi > 1 \) and \( \psi > 1 \) as in BY, the above proposition implies that the low-frequency covariation between consumption and excess returns is higher than the high-frequency covariation. Therefore, the one-channel long-run risk model can qualitatively match the observed patterns in the long-run correlation and lead-lag relation between consumption and excess returns, even the model implies a constant risk premium. The reason that the one-channel model can produce the desired pattern is that the innovation in the excess return has a cyclical consumption component, \( e_{t+1} \) (see Eq. 13). Thus, past returns can positively predict future growth. Since excess asset returns are i.i.d, this positive covariance between past returns and future consumption growth also leads to a higher long-run correlation between consumption and asset returns.

The IES parameter \( \psi \) is very important here. If the IES \( \psi \) is smaller than \( 1/\phi \), then the cospectrum is increasing and asset returns negatively predict future consumption growth, both of which are the opposite of that found in the data. BY show that, to ensure that the price–dividend ratio is positively related to expected consumption growth, the condition \( \psi > 1/\phi \) is also required. Thus, the condition \( \psi > 1/\phi \) is crucial for many important features of the BY model.

BY show that the one-channel model can produce a sizable equity premium. However, the risk premium is approximately a constant in this one-channel model, whereas many studies argue that the risk premium is time varying. To produce this feature, BY introduce a two-channel model with both predictable consumption and stochastic consumption volatility. I briefly discuss the long-run properties of the BY two-channel model below.

**Two-channel long-run risk model** Following BY, in this example, the dynamics of consumption and dividends are assumed to be

\[
\begin{align*}
  x_{t+1} &= \rho x_t + \phi \eta_t e_{t+1}, \\
  g_{c,t+1} &= \mu + x_t + \sigma_t \eta_t e_{t+1}, \\
  g_{d,t+1} &= \mu_d + x_t + \phi \eta_t u_{t+1}, \\
  \sigma_{x,t}^2 &= \sigma^2 + v_1 \{ \sigma^2 - \sigma^2 \} + \sigma_w w_{t+1}, \\
  \sigma_{e,t}^2 &= \sigma_w^2.
\end{align*}
\]

(15)

where the innovations \( e_t, \eta_t, u_t, \) and \( w_t \) are i.i.d. normal, and all the innovations are independent of each other, except \( \text{corr}(\eta_t, u_t) > 0 \). It follows from a similar log-linearization argument as in Bansal and Yaron (2004) that the excess returns can be approximated by

\[
\begin{align*}
  r_{t+1}^e &\approx \beta_{m,w} \lambda_{m,w} \sigma_w^2 - 0.5 \beta_{m,w}^2 \sigma_w^2 + (\beta_{m,e} \lambda_{m,e} - 0.5 \beta_{m,e}^2 - 0.5 \phi_{d}) \sigma_t^2 \\
  &+ \kappa_{1,m} A_{1,m} \phi_e \sigma_t e_{t+1} + \kappa_{1,m} A_{2,m} \sigma_w w_{t+1} + \kappa_{1,m} A_{2,m} \sigma_t u_{t+1},
\end{align*}
\]

(16)

where all the constants \( \beta_{m,w}, \beta_{m,e}, \lambda_{m,w}, A_{1,m} \) and \( A_{2,m} \) are just functions of the primitive parameters. Thus, in this model the conditional equity premium is given by

\[
E_t r_{t+1}^e = (\beta_{m,e} \lambda_{m,e} - 0.5 \beta_{m,e}^2 - 0.5 \phi_{d}) \sigma_t^2.
\]

(17)

Thus, the equity premium is predictable in this model, the same as that in the data.

Similar to the one-channel model, the innovation component \( \kappa_{1,m} A_{1,m} \phi_e \sigma_t e_{t+1} \) in Eq. (16) can predict future consumption growth. Since all the other components in Eq. (16) are independent of the cyclical consumption component \( x_t \), the two-channel model also implies that excess returns positively predict future consumption growth, just as in the one-channel model. Thus, decreasing spectra are also expected as before. However, for this two-channel long-run risk model, analytical results for the spectrum cannot be obtained. Thus, I simulate the model in the next section and quantitatively study its implications for the joint dynamics of consumption and asset returns.
Finally, for the two-channel long-run risk model, if one replaces $x_{t+1} = \rho x_t + \varphi \sigma_t e_{t+1}$ in Eq. (15) with
\[ x_{t+1} = \rho x_t + \varphi \sigma_t^2 - \sigma_t^2 + \varphi \sigma_t e_{t+1}, \]
the extended two-channel model can produce a countercyclical equity premium (see Eq. (17)). Here, the additional parameter $\varphi \sigma_t^2 - \sigma_t^2$ captures the notion of countercyclical volatility. That is, high consumption volatility is usually associated with the times when consumption is below its stochastic trend and hence has a higher subsequent expected growth rate. The above specification for consumption is a special case of Backus et al. (2010), who show that this specification can match the frequency for the short rate and excess returns to lead the business cycle in the data. In the Internet Appendix, I show that the countercyclical expected returns can help produce a decreasing pattern in the spectrum. In addition, the general equilibrium model of Garleanu et al. (forthcoming) provides such an example.

It is worth noting that to match the lead-lag relation between excess stock market returns and consumption growth, countercyclical consumption volatility is not required, as discussed in Section 3.2. Indeed, panel d of Fig. 12 in Backus et al. (2010) also shows that asset returns lead consumption growth even when consumption volatility is acyclical. However, Backus et al. (2010) show that countercyclical volatility is needed to match the cross correlation between the short rate and consumption. Moreover, Backus et al. (2010) show that countercyclical volatility can strengthen the already correct lead-lag relation between excess stock market returns and consumption. Thus, the countercyclical consumption volatility in the BY model should further help to produce a higher correlation between consumption and excess returns at low frequencies than at high frequencies.

In sum, I show above that models that feature a cyclical component in consumption, and that link returns to this cyclical component, are particularly good at matching the long-run features in the data. This is intuitive. To see the lead-lag relation in this type of model, assume that the realized asset return is high in the current period. It is likely that there is a positive shock to expected consumption growth, and hence future consumption growth is anticipated to be high. Thus, high asset returns anticipate a high consumption growth rate. Since the growth rate is persistent, this positive association between returns and the future growth rate leads to higher long-run consumption-return correlations.

4. A quantitative examination

The previous section uses analytical approximations to show that for reasonable calibrations, the habit formation model cannot produce the lead-lag relation and the long-run correlations between consumption and asset returns, whereas the long-run risk model can qualitatively match the observed patterns. In this section, I evaluate the model performance quantitatively using numerical methods.

4.1. External habit models

4.1.1. Parameter habit models

The parameter values are chosen to be the same as those in CC whenever possible. These values are reported in panel A of Table 1. The approximation analysis in the previous section provides good intuition on the qualitative features of the model in the long run. To obtain the quantitative features of the model and to compare these features with the data, I solve this model numerically by the method developed by Wachter (2005). To obtain fast convergence and high precision, Wachter (2005) computes the price–dividend ratio as a series of “zero-coupon equity” claims. Since the cointegration constraint is incorporated into the model, the persistence parameter $\phi_\delta$ for the difference between log dividends and log consumption needs to be chosen. This parameter is taken from Bansal et al. (2007). For comparison with CC, I also report results for the case $\phi_\delta = 1$ and the correlation between consumption growth and dividend growth $\rho_{c,d} = 0.2$ (i.e., model CC2). The parameterizations are exactly the same as in CC.

I simulate 200,000 quarters of artificial data to calculate population values for a variety of statistics for each model. Panel A of Table 1 reports the parameter values for each model. Panel B shows the summary statistics of the equity premium, the risk-free rate, and the price–dividend ratio from the simulated model. To facilitate comparison with CC, I report the simulated moments of consumption and asset returns together with those of both the postwar sample and the long sample from Table 2 of CC. As in CC, the external habit formation model matches these moments well. Consistent with Wachter (2005), when the model is solved with higher precision, the equity premium is slightly lower than that originally reported by CC.

4.1.2. Long-run correlations

To see whether the implied long-run features of the model are consistent with the data, I plot the correlations between consumption growth and asset returns at different horizons in Fig. 2. In the data, the correlation between consumption and returns is upward sloping. However, for the habit formation model, the correlation is monotonically decreasing in the horizon as predicted by the log-linear approximation.

I consider three variants of the external habit model. Habit model 1 (CC1) is a baseline model where consumption and dividends are cointegrated. In habit model 2 (CC2), I set free the cointegration constraint. That is, I set $\phi_\delta = 1$. The parameter values in both models are shown in panel A of Table 1. When consumption and dividends are not cointegrated as in CC,
the correlation between consumption and dividends is indeed lower, as shown in Fig. 2. However, the decreasing pattern is still there. The correlation between consumption growth and asset returns is a bit too large in the model. However, this correlation level can be lowered with a different parameter configuration, as shown by habit model 3. Habit model 3 uses a lower correlation between innovations in consumption and dividends (e.g., $\rho_{cd} = 0.08$, and $\sigma_d = 0.15$), and the model generates a sizable 3% equity premium with risk aversion $\gamma = 2$. By contrast, the decreasing pattern in the correlation over the long horizon remains.

### 4.1.3. Spectral analysis

Propositions 1 and 2 in Section 3.1 provide the qualitative features of the cross spectrum between consumption and asset returns by a log-linear approximation for the baseline habit model. The exact cross spectra can be obtained from 200,000 quarters of artificial data with the parameter values given by model CC1 in panel A of Table 1. The top panel of Fig. 3 plots the coherence between consumption growth and asset returns from the model simulation, and the middle panel plots the cospectrum. The cospectrum is normalized so that the area under the curve is one. As mentioned earlier, this makes the comparisons among different models more transparent, since the shape of the cospectrum is the focus of this paper. The bottom panel is the phase spectrum, which is increasing. The dotted (bumpy) line in Fig. 3 shows that in the simulated model, the cospectrum is increasing. The solid (smooth) line is the cospectrum from the analytical approximation. The approximation is quite accurate (at least for the purpose of the cross spectrum) even though the model is nonlinear. The difference between the linear approximation and the exact solution is generally small, and the shape of the spectrum is very similar. Again, the inverse of the frequency measures the number of quarters for the cycle at that frequency. For example, when $\lambda = 0.1$, the length of the cycle is 10 quarters. For model CC2, the shape of the spectra remains similar.

Taken together, Figs. 1 and 3 show that the coherence, cospectrum, and phase spectrum are all declining in the data, whereas they are all increasing in the external habit formation models. In other words, in the data, asset market returns lead consumption and more covariation between consumption and returns comes from low frequencies, whereas the opposite is true for the external habit model.

### 4.1.4. Robustness check: habit model with predictable consumption

In the benchmark model, I follow the original setup of CC by imposing i.i.d. consumption growth. It has been shown that the external habit model with i.i.d. consumption growth cannot produce a consistent cross spectrum between consumption and asset returns as seen in the data. Since a cyclical component in consumption is the key assumption in the BY model, it is important to make sure that the implications in the previous section hold when consumption is predictable in the habit model. Therefore, I resolve the model in Section 3.1.1 by replacing i.i.d. consumption growth in Eq. (1) with an ARMA(2, 2)

### Table 1

**External habit models with i.i.d. growth rates.**

Panel A: Parameter choices

<table>
<thead>
<tr>
<th>Statistics</th>
<th>Variable</th>
<th>Model CC1</th>
<th>Model CC2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean consumption growth (%)</td>
<td>$\mu_c$</td>
<td>1.89</td>
<td>1.89</td>
</tr>
<tr>
<td>Standard deviation of consumption growth (%)</td>
<td>$\sigma_c$</td>
<td>1.50</td>
<td>1.50</td>
</tr>
<tr>
<td>Log risk-free rate (%)</td>
<td>$r_f$</td>
<td>0.94</td>
<td>0.94</td>
</tr>
<tr>
<td>Persistence coefficient in habit</td>
<td>$\phi_h$</td>
<td>0.87</td>
<td>0.87</td>
</tr>
<tr>
<td>Persistence coefficient in $\delta_t$</td>
<td>$\rho_\delta$</td>
<td>0.89</td>
<td>1</td>
</tr>
<tr>
<td>Standard deviation of the innovation in $\delta_t$</td>
<td>$\sigma_\delta$</td>
<td>0.112</td>
<td>0.112</td>
</tr>
<tr>
<td>Risk aversion coefficient</td>
<td>$\gamma$</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Subjective discount factor</td>
<td>$\beta$</td>
<td>0.89</td>
<td>0.89</td>
</tr>
<tr>
<td>Correlation between $g_{c,t}$ and $s_{d,t}$</td>
<td>$\rho_{cd}$</td>
<td>−0.1</td>
<td>NA</td>
</tr>
<tr>
<td>Correlation between $g_{c,t}$ and $g_{d,t}$</td>
<td>$\rho_{cd}$</td>
<td>0.2</td>
<td></td>
</tr>
</tbody>
</table>

Panel B: Summary statistics from simulation

<table>
<thead>
<tr>
<th>Statistics</th>
<th>Variable</th>
<th>Model CC1</th>
<th>Model CC2</th>
<th>Postwar sample</th>
<th>Long sample</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\rho = 0.89, \rho_{cd} = 0.1$</td>
<td>$\rho = 1$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E(g_{c,t})$ (%)</td>
<td>1.90</td>
<td>1.89</td>
<td>1.89</td>
<td>1.72</td>
<td></td>
</tr>
<tr>
<td>$\sigma(g_{c,t})$ (%)</td>
<td>1.50</td>
<td>1.50</td>
<td>1.22</td>
<td>3.32</td>
<td></td>
</tr>
<tr>
<td>$E(r_{f})$ (%)</td>
<td>0.94</td>
<td>0.94</td>
<td>0.94</td>
<td>2.92</td>
<td></td>
</tr>
<tr>
<td>$E(r - r_{f})$ (%)</td>
<td>3.58</td>
<td>4.27</td>
<td>6.69</td>
<td>3.90</td>
<td></td>
</tr>
<tr>
<td>$\sigma(r - r_{f})$ (%)</td>
<td>8.33</td>
<td>15.54</td>
<td>15.7</td>
<td>18.0</td>
<td></td>
</tr>
<tr>
<td>$\exp[E(p - d)]$</td>
<td>39.08</td>
<td>30.94</td>
<td>24.7</td>
<td>21.1</td>
<td></td>
</tr>
<tr>
<td>$\sigma(p - d)$</td>
<td>0.23</td>
<td>0.16</td>
<td>0.26</td>
<td>0.27</td>
<td></td>
</tr>
<tr>
<td>$AC_1(p - d)$</td>
<td>0.88</td>
<td>0.86</td>
<td>0.87</td>
<td>0.78</td>
<td></td>
</tr>
</tbody>
</table>

Panel A reports the parameter choices for the external habit formation model with i.i.d. consumption growth. All the parameter values are annualized as in CC. Model CC1 is the baseline model with cointegrated consumption and dividends. Model CC2 imposes no cointegration. It assumes that growth rates for both consumption and dividends are i.i.d. and that the correlation between consumption innovations and dividend innovations is 0.2. Panel B reports summary statistics of simulated data for external habit formation models with i.i.d. consumption growth for both models CC1 and CC2. For each model, 200,000 quarters of artificial data are simulated based on parameter choices in panel A. The models are solved by the series method in Wachter (2005). The quantities for the data are taken directly from Campbell and Cochrane (1999)’s Table 2 for ease of comparison.
Fig. 2. Long-run correlation between consumption and excess returns. This figure plots the correlations between consumption growth and returns at different horizons for three different habit models and the data. Habit model 1 is the calibrated model CC1 in Table 1. Habit model 2 is the model without cointegration between consumption and dividends, and is the model CC2 in Table 1. Habit model 3 is the habit model with a lower correlation between consumption and dividends ($\rho_{c,d} = 0.08$). The data line is for the correlation between consumption and excess returns at different horizons for the data.

Quarterly real seasonally adjusted consumption per capita from 1947Q1 to 2009Q4 is obtained from the BEA. Consumption is for nondurables and services excluding shoes and clothing. The corresponding quarterly excess value-weighted market return is taken from the CRSP VW index.

process. The parameter values for the consumption dynamics are estimated with the historical quarterly consumption data and are reported in the first part of panel A in Table 2. Using these parameter values, I solve this model numerically. This model produces a similarly large equity premium and low risk-free rate as before. Moreover, the correlation is still decreasing as the horizon increases. Panel B of Table 2 reports the correlation between consumption and asset returns at different horizons. The correlation decreases from 73% to 41% as the horizon increases from 1 to 20 quarters.

To understand why the results still hold in the case with predictable consumption growth, first notice that the surplus ratio is the effective risk aversion, and hence the excess returns can still be written as $E_t(\rho_{x,t+1}) \approx \alpha - \beta_5 \sum_{j=1}^{\infty} \phi_j^{j-1} g_{c,t+1-j}$. Thus, the model is still backward looking, and past consumption growth negatively forecasts future returns, leading to a lower long-run consumption-return correlation, just as before. Accordingly, the external habit persistence model with predictable consumption growth cannot produce the desired lead-lag relation and a higher low-frequency correlation between consumption and returns because the main driving force in the model is still the surplus ratio.

4.2. Long-run risk models

I also consider three versions of the long-run risk models. The first is the original one-channel BY model with zero correlation between consumption and dividend innovations. The second BY model allows for a positive correlation between

---

6 If I use the same predictable consumption process as that in the BY model, the results remain very similar to the i.i.d. consumption growth case, since the BY consumption process is very close to i.i.d. By using a more predictable consumption growth rate than that in the BY model, I try to increase the habit model’s ability to replicate the patterns in the data.
consumption innovations and dividend innovations. The last BY model is the two-channel BY model with stochastic volatility. Table 3 reports the parameter values for these three BY models. It is well known that these models can match the first two moments of excess market returns and risk-free rates. Since the parameter choices are almost identical to those in BY, to save space, I do not report the first two moments of these models.

For the first BY model without stochastic volatility in consumption, the left three panels of Fig. 4 plot the analytical coherence, raw cospectrum, and phase spectrum between consumption growth and excess returns for model BY0 in Table 3. The decreasing pattern is the same as that in the data. Under the assumption $\rho_{u}\kappa = \text{corr}(\eta_t, u_t) = 0$ as in the original BY one-channel model, the covariance between excess returns and consumption growth is extremely small. Indeed, Eqs. (12) and (13) imply that $\text{cov}(r_{ex}^t, g_t) \approx 0$. Thus, I only report the raw cospectrum, since the normalized cospectrum has no meaning when the determinator is zero. In the right three panels of Fig. 4, I allow a plausible positive correlation between innovation in consumption and innovation in dividend to be 0.2. As shown, both the coherence and the cospectrum are still decreasing at all frequencies. In addition, the normalized cospectrum has a magnitude similar to that in the data. The phase spectrum is decreasing at very low frequencies, then increasing at higher frequencies. Since the covariations at low frequencies are much higher than at higher frequencies, the low frequencies carry more weight. Overall, the phase spectrum indicates that asset returns lead consumption. Indeed, it is true that $\text{cov}(r_{ex}^{t+1}, g_{t+k}) = \frac{\delta^{-1} - \sigma^2}{1 - \kappa_1 \rho \kappa_1 m \rho^k - 2 \psi \sigma^2} > 0$ for $k > 1$ and IES $\psi > 1/\phi$. Thus, asset returns positively lead to consumption growth.

---

**Fig. 3.** Cross spectra for the simulated data and analytical approximation. The three panels in this figure plot the coherence, normalized cospectrum, and phase spectrum between consumption growth and excess stock market returns in the simulated model. The parameter values are given by model CC1 in panel A of Table 1. The normalized cospectrum is calculated such that the area under the curve is one. The dotted (smooth) line is calculated from the analytical approximation, and the solid (bumpy) line is calculated from the 200,000 quarters of simulated data.
correlation and lead-lag relation between consumption and excess returns. Hence, a large IES is required to produce many important features of the long-run risk model. In sum, the above analysis shows that for large IES, the one-channel long-run risk model can reproduce the observed patterns in the long-run risk model.

Other parameters:
- Persistence coefficient in habit: \( \phi \)
- Persistence coefficient in \( \delta \): \( \rho_\delta \)
- Standard deviation of the innovation in \( \delta_t \): \( \sigma_\delta \)
- Risk aversion coefficient: \( \gamma \)
- Correlation between \( \epsilon_{t, 1} \) and \( \epsilon_{t, 2} \): \( \rho_{\epsilon, \delta} \)
- Subjective discount factor: \( \beta \)

Panel B: Long-horizon correlation between consumption and excess returns

<table>
<thead>
<tr>
<th>Horizon (in quarters)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>8</th>
<th>12</th>
<th>16</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cumulative correlation</td>
<td>0.73</td>
<td>0.67</td>
<td>0.63</td>
<td>0.61</td>
<td>0.56</td>
<td>0.51</td>
<td>0.46</td>
<td>0.41</td>
</tr>
</tbody>
</table>

Panel A reports parameter choices for the external habit formation model with ARMA(2, 2) consumption growth. Since quarterly consumption data are used to estimate the consumption dynamics as input of the model, in this table all parameter values are at a quarterly frequency. Quarterly real seasonally adjusted consumption per capita from 1947Q1 to 2009Q4 is obtained from the BEA. Consumption is for nondurables and services excluding shoes and clothing. Panel B reports the cumulative correlations between consumption and excess returns, which are decreasing as horizons increase.

### Table 2

External habit model with predictable growth rates.

#### Panel A: Parameter choices

<table>
<thead>
<tr>
<th>Statistics</th>
<th>Variable</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Consumption dynamics:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean consumption growth (%)</td>
<td>( \mu_c )</td>
<td>0.4575</td>
</tr>
<tr>
<td>Standard deviation of the innovation in consumption (%)</td>
<td>( \sigma_c )</td>
<td>0.4841</td>
</tr>
<tr>
<td>AR(1) coefficient of consumption growth</td>
<td>( \varphi_{t, 1} )</td>
<td>0.9963</td>
</tr>
<tr>
<td>AR(2) coefficient of consumption growth</td>
<td>( \theta_{t, 2} )</td>
<td>-0.2507</td>
</tr>
<tr>
<td>MA(1) coefficient of consumption growth</td>
<td>( \theta_{t, 1} )</td>
<td>-0.8302</td>
</tr>
<tr>
<td>MA(2) coefficient of consumption growth</td>
<td>( \theta_{t, 2} )</td>
<td>0.3069</td>
</tr>
<tr>
<td>Other parameters:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Persistence coefficient in habit</td>
<td>( \phi )</td>
<td>0.9658</td>
</tr>
<tr>
<td>Persistence coefficient in ( \delta )</td>
<td>( \rho_\delta )</td>
<td>0.9719</td>
</tr>
<tr>
<td>Standard deviation of the innovation in ( \delta_t )</td>
<td>( \sigma_\delta )</td>
<td>0.056</td>
</tr>
<tr>
<td>Risk aversion coefficient</td>
<td>( \gamma )</td>
<td>1.5</td>
</tr>
<tr>
<td>Correlation between ( \epsilon_{t, 1} ) and ( \epsilon_{t, 2} )</td>
<td>( \rho_{\epsilon, \delta} )</td>
<td>-0.05</td>
</tr>
<tr>
<td>Subjective discount factor</td>
<td>( \beta )</td>
<td>0.9740</td>
</tr>
</tbody>
</table>

#### Panel B: Parameter choices for long-run risk models

<table>
<thead>
<tr>
<th>Statistics</th>
<th>Variable</th>
<th>BY0</th>
<th>BY1</th>
<th>BY2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean consumption growth (%)</td>
<td>( \mu_c )</td>
<td>0.15</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean dividend growth (%)</td>
<td>( \mu_d )</td>
<td>0.15</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Standard deviation of consumption innovation (%)</td>
<td>( \sigma )</td>
<td>0.78</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Persistence coefficient in expected growth</td>
<td>( \rho )</td>
<td>0.979</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dividend exposure to long-run consumption</td>
<td>( \phi )</td>
<td>3.3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Long-run consumption volatility</td>
<td>( \psi_c )</td>
<td>0.044</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Volatility of dividend innovations</td>
<td>( \psi_d )</td>
<td>4.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Risk aversion coefficient</td>
<td>( \gamma )</td>
<td>10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>IES coefficient</td>
<td>( \psi )</td>
<td>1.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Subjective discount factor</td>
<td>( \delta )</td>
<td>0.998</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Volatility of volatility (( x 10^{-5} ))</td>
<td>( \sigma_{\epsilon} )</td>
<td>0</td>
<td></td>
<td>0.23</td>
</tr>
<tr>
<td>Persistence of consumption volatility</td>
<td>( \gamma_1 )</td>
<td>0.987</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Correlation b/t consumption and dividend innovations</td>
<td>( \rho_{\epsilon, \delta} )</td>
<td>0</td>
<td>0.2</td>
<td>0.2</td>
</tr>
</tbody>
</table>

This table reports the parameter choices for the long-run risk models. The parameter choices are almost the same as in Bansal and Yaron’s (2004) original one-channel and two-channel models. The only difference is that in models BY1 and BY2, a positive correlation between consumption innovations and dividend innovations is allowed. As in Bansal and Yaron (2004), these parameter values are at a monthly frequency. The simulated data for consumption growth and excess stock market returns are then time aggregated to quarterly and annual frequencies. Model BY0 is the original one-channel BY model. Model BY1 is the same as BY0 except for a positive correlation between consumption innovations and dividend innovations. Model BY2 introduces stochastic consumption volatility on top of model BY1.

As mentioned earlier, the IES is crucial to producing the decreasing pattern in the cospectrum. Fig. 5 plots the cross spectra for a BY model with an IES \( \psi = 0.3 < 1/\phi \). As predicted by Proposition 3, the cospectrum is indeed decreasing. Hence, a large IES is required to produce many important features of the long-run risk model. In sum, the above analysis indicates that for large IES, the one-channel long-run risk model can reproduce the observed patterns in the long-run correlation and lead-lag relation between consumption and excess returns.\(^7\)

Now I turn to the two-channel long-run risk model BY2. The left three panels in Fig. 6 show that for simulated quarterly data, the coherence, cospectrum, and phase spectrum are all decreasing. Therefore, the long-run risk model produces a higher comovement between consumption and returns over longer horizons, and the model also implies that asset returns lead consumption. Compared with the one-channel long-run risk model, the two-channel model produces a less steep slope.

\(^7\) Slightly overshooting occurs at the extremely low frequencies in Fig. 4. This overshooting is due to the extremely high persistence in the expected growth rate. One solution is to allow for a less persistent and more volatile expected growth rate. This way, the equity premium can remain large and at the same time, the cospectrum will be less steep.
Fig. 4. Cross spectra for one-channel long-run risk model. The left three panels plot the analytical coherence, raw cospectrum, and phase spectrum between consumption growth and excess market returns for the one-channel long-run risk model of Bansal and Yaron (2004). The parameter values are given by model BY0 in Table 3. The right three panels plot the analytical coherence, normalized cospectrum, and phase spectrum between consumption growth and excess market returns for the one-channel long-run risk model BY1 in Table 3. The normalized cospectrum is calculated such that the area under the curve is one.

For the cospectrum at low frequencies, which is more consistent with the data. Similar to the data, the phase spectrum at high frequencies is very volatile. This is again due to the low coherence at high frequencies. When the comovement is weak, it is hard to determine which series are leading and which are lagging. The right three panels in Fig. 6 plot the cross spectra for annual simulated data. The decreasing shape still remains, and the slopes are less steep than those for the quarterly simulated data, consistent with the pattern in the real quarterly and annual data.

In sum, with a large IES, the BY model can reproduce the decreasing pattern in the cross spectrum between consumption and asset returns. Meanwhile, the model can also match the first two moments of asset prices and the predictability of excess returns.

4.3. Potential small-sample issue

To verify the coherence results, I also conduct a time-series band-pass filtering analysis. Moreover, to account for potential small-sample issues, I also run 1000 Monte Carlo simulations for each model, each with 400 quarters of observations. I then use a band-pass filter to calculate the low-frequency (with cycles longer than three years) and high-frequency (with cycles between 0.5 and three years) correlations between consumption and asset returns in each Monte Carlo experiment. Then, the difference between the low-frequency and high-frequency correlations is obtained for each experiment. In the
Fig. 5. Cross spectra for one-channel long-run risk model with low IES. The three panels plot the analytical coherence, normalized cospectrum, and phase spectrum between consumption growth and excess market returns for the one-channel long-run risk model of Bansal and Yaron (2004). The parameter values are the same as in model BY1 in Table 3, except with a lower IES $\psi = 0.3$.

data, the difference between the low-frequency and high-frequency correlations is about 25%. For the habit models, none of the Monte Carlo experiments for the external habit model can produce such a significant difference. Hence, the habit models cannot produce the same long-horizon features as those in the data. Fig. 7 plots the histogram for the differences between the low-frequency and high-frequency correlations in the consumption and asset returns from the Monte Carlo experiments. Indeed, the top two panels indicate that for the habit model, all the differences are less than the difference in the data, and most of these differences are negative as predicted by the habit model. Thus, the implied $p$-value from this Monte Carlo exercise is zero.

I perform the same Monte Carlo analysis for models BY1 and BY2. The results are presented in the bottom two panels in Fig. 7. It is evident that the 25% value from the data lies well within the 95% confidence interval derived from the simulated empirical distributions. Thus, both BY1 and BY2 can produce a large positive difference in the correlations between low and high frequencies.

Finally, Table 4 provides a summary of the main results for different models in the most intuitive form. It reports the results based on band-pass filter analysis, Granger causality tests and simple cumulative correlation analysis. The results from both a long-sample simulation and repeated small-sample simulations are reported.\(^8\) Both band-pass filter analysis and simple cumulative correlation analysis confirm the spectral analysis that the habit model produces lower long-run

\(^8\) The 95% confidence intervals for the Granger causality tests from small-sample simulations are extremely wide for all the models. Thus, to save space, these results, which are not very informative, are not reported.
Fig. 6. Cross spectra for the long-run risk model with stochastic volatility. The left (right) three panels plot the coherence, normalized cospectrum, and phase spectrum between the quarterly (annual) consumption growth rate and the quarterly (annual) excess stock market returns for the simulated data in a calibrated model of Bansal and Yaron (2004). The normalized cospectrum is calculated such that the area under the curve is one. The parameters are given by model BY2 in Table 3. The graph indicates that returns lead consumption and comovements between consumption and returns are stronger at low frequencies.

In sum, I show that different versions of models with predictable consumption can match the long-run and lead-lag relation between consumption and excess returns, as well as the equity premium. Here, I do not claim that these models are better than habit formation models. I simply use these models as examples to illustrate the ingredients that are necessary to reproduce the key patterns in the joint consumption-return dynamics.

5. General lessons

In this section, I try to draw general lessons from the above analytical and numerical analysis. In particular, I highlight the necessary ingredients for a model to produce the observed patterns in joint consumption-return dynamics. Obviously, the first necessary condition is that (1) the log consumption should include both a stochastic trend and a cyclical component. That is, consumption growth should be predictable. The second tautological condition is that (2) the realized return should
Fig. 7. Histogram for small-sample simulation. This figure plots the histogram for the differences between the low-frequency and high-frequency correlations in consumption growth and asset returns from 1000 Monte Carlo simulations for the external habit formation models CC1 and CC2, and the long-run risk models BY1 and BY2. Each simulation has a sample of 400 quarters (100 years) of observations. The straight line is the difference between the low-frequency and the high-frequency correlation in consumption growth and asset returns for the historical data. Quarterly real seasonally adjusted consumption per capita is obtained from the BEA. Consumption is for nondurables and services excluding shoes and clothing.

be (negatively) linked to the cyclical consumption component and is thus positively associated with future consumption growth. Thus, asset returns lead consumption growth.

A model can satisfy the second condition in two alternative ways. First, (2a) the innovation in the realized return is negatively linked to the cyclical component of consumption (thus, it positively predicts future growth rates). The one-channel BY model provides such an example. In this model, the expected equity premium is a constant, and the innovation in excess returns is directly linked to the cyclical consumption component. Second, (2b) the model-implied expected excess return is positively linked to the cyclical component of consumption. The two-channel BY model with countercyclical consumption volatility and the model by Garleanu et al. (forthcoming) provide such examples, since expected asset returns are linked to anticipated consumption growth. Given the fact that excess returns are predictable and the cyclical component in output and consumption can strongly forecast both asset returns (see, e.g., Cooper and Priestley, 2009), an area for future research would be to develop asset pricing models linking the cyclical component in output/consumption to expected returns (i.e., condition (2b)).

Finally, an additional ingredient needed is that (3) the innovation in the cyclical component and the innovation in the trend component should be negatively correlated, as shown in the proof for Proposition 3.9 Moreover, as shown in the

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9 For Proposition 3, the zero correlation between innovation in the growth rate ($\eta_t$) and innovation in the expected growth rate ($\epsilon_t$) implicitly implies a negative correlation between the innovation in the cycle component and the innovation in the trend component (see Internet Appendix for details).
In this paper, I analyze asset pricing models by focusing on their low-frequency implications. I argue that the standard external habit formation model faces a challenge in generating the same long-run correlation and lead-lag relation between consumption and market returns as in the data. However, different versions of the long-run risk models can generate the same patterns as those found in the data. In these models, log consumption includes both a stochastic trend and a cyclical component. Moreover, expected returns or innovations in asset returns depend negatively on the cyclical consumption component. I conclude that forward-looking consumption components in the pricing of risk are important in matching these features of the data. In this sense, my study complements that of Backus et al. (2010), which focuses on the lead-lag relation.

6. Conclusions

In this paper, I analyze asset pricing models by focusing on their low-frequency implications. I argue that the standard external habit formation model faces a challenge in generating the same long-run correlation and lead-lag relation between consumption and market returns as in the data. However, different versions of the long-run risk models can generate the same patterns as those found in the data. In these models, log consumption includes both a stochastic trend and a cyclical component. Moreover, expected returns or innovations in asset returns depend negatively on the cyclical consumption component. I conclude that forward-looking consumption components in the pricing of risk are important in matching these features of the data.

Supplementary material

The online version of this article contains additional supplementary material.
Please visit http://dx.doi.org/10.1016/j.red.2012.04.001.
References


