Optimal Equity Auctions When Bidders Are Ex-Ante Heterogeneous

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Abstract

I analyze the effects of heterogeneous bidders who differ in sizes and distributions of synergy gains in equity auctions. Such heterogeneity affects the ordering of equity bids, rendering the seller’s revenues sensitive to the auction design. Among all incentive-compatible mechanisms, I identify the optimal mechanism that maximizes the seller’s expected revenues. I show how bidder heterogeneity alters the optimal auction design, and obtain the distinct implications of different sources of bidder heterogeneity. I provide intuitive characterizations of the optimal mechanism’s properties; I also show the optimal mechanism has features with implications that go beyond the direct first-order intuition.

Key words: Auctions; Mechanism design; Equity bids; Bidder heterogeneity

JEL classification: D44; D82

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1 Introduction

In many auctions, rather than pay cash, bidders pay with securities whose value depends on the realization of future cash flows generated by the auctioned asset or project. The most common type of security auctions is equity auctions, in which bidders pay with fractions of the total cash flow. Equity auctions arise in a wide variety of economic situations. Andrade, Mitchell, and Stafford (2001) document an overwhelming use of equity payments in mergers and acquisitions, with 58% of the transactions paid entirely in equity and 70% involving equity. Skrzypacz (2013) reports oil and gas lease auctions typically feature equity payments in the form of royalties. Project procurement auctions, venture capitalist financing, and lead-plaintiff auctions also widely use equities as the means of payment.

The literature on equity auctions has focused on ex-ante identical bidders, where bidders have the same standalone market values, investment costs, and synergy distributions. In their seminal papers, Hansen (1985), Cremer (1987), and Riley (1988) demonstrate the advantages of equity over cash bids, and DeMarzo, Kremer, and Skrzypacz (2005) derive a very elegant irrelevance result that all standard formats (e.g., first-price and second-price formats) of equity auctions yield the same expected revenue when bidders are ex-ante identical.

While the assumption of ex-ante identical bidders simplifies the analysis and provides important insights into the working of equity auctions, it is important in practice to understand the impact of ex-ante bidder heterogeneity. Such heterogeneity is a common feature of many auction settings: bidders often differ in their observable characteristics such as size or distribution of valuations. For example, in takeover auctions, some bidders may have much higher market values than others. In auctions for project rights, bidders may face different opportunity costs or financing costs.

Ex-ante bidder heterogeneity has important relevance for equity auctions. Unlike cash bids, the monetary values of equity bids are not transparent. They depend on the bids’ face values (i.e., equity shares), bidders’ observable characteristics (e.g., market values), and bidders’ private types (e.g., synergies). When bidders are ex-ante identical and employ symmetric strategies, bids’ face values, monetary values, and bidder types align: higher-type bidders submit bids with higher face and monetary values. This alignment simplifies the selection of the winning bid. In sharp contrast, when bidders differ ex ante, the equilibria will no longer be symmetric. Consequently,
the aforementioned alignment breaks down, and the bid ranking becomes ambiguous. Reflecting this ambiguity, bidder heterogeneity directly affects the performance of equity auctions. Whereas the seller’s expected revenue is insensitive to the auction design and standard formats always generate higher revenues than cash auctions in the absence of bidder heterogeneity, when bidders differ ex ante, expected revenues for different auction formats can vary widely, and hinge sensitively on the nature of the heterogeneity. Moreover, standard formats can generate lower revenues than cash auctions. Consequently, the proper design for equity auctions becomes important with ex-ante heterogeneous bidders.

In this paper, I investigate the design of equity auctions with ex-ante heterogeneous bidders. To the best of my knowledge, my paper is the first to do so. Given bidder heterogeneity, I identify the selling mechanism that maximizes the seller’s expected revenues among all incentive compatible mechanisms. I show how bidder heterogeneity alters the optimal design, and derive the distinct implications of different sources of bidder heterogeneity. I provide intuitive characterizations of the properties of the optimal design, and I show the optimal design also exhibits features that have deeper implications beyond direct first-order intuition.

In my model, risk-neutral bidders bid for a target in a takeover auction in which they (winner and losers) pay with equities. Bidders privately observe their synergies with the target, and synergies are distributed independently. The bidders’ and target’s standalone market values and the synergy distributions are common knowledge. I incorporate two important sources of ex-ante bidder heterogeneity: bidders may have different market values and their synergy distributions may differ. Although I place the model in the context of takeover auctions, the results apply generally to the sale of any indivisible asset through equity payments.

I utilize an adaptation of the classic mechanism-design approach for cash auctions (Myerson 1981) to solve for optimal equity auctions. I determine their allocative properties (who wins) and paying features (which bidders pay and the amounts). I show optimal equity auctions have the property that losing bidders never pay. To understand this result, note the incentive compatibility constraint implies bidders earn informational rents for having private synergies, which they can consume upon winning. If bidders do not retain the full extent of their synergies when they win, as in equity auctions, their informational advantages—hence their rents—scale accordingly. If bidders pay upon losing, their payments upon winning typically reduce, raising
the winner’s retained equity share and thus bidders’ informational advantages, and hence decreasing the seller’s revenue. Thus, it is optimal to have only the winner pay. This result contrasts with cash auctions, in which mechanisms requiring losing bidders to pay (e.g., all-pay auctions) can also be optimal, because a bidder in cash auctions retains all of its equity upon winning (thereby retaining the full extent of its informational advantage), whether or not it would pay upon losing.

When bidders are ex-ante identical, optimal equity auctions are simple: standard (e.g., first- and second-price) equity auctions with an optimal reserve price are optimal.\(^1\) This result is similar to the result in cash auctions that shows standard formats with an optimal reserve price are optimal when bidders have the same synergy distribution. In both optimal equity and optimal cash auctions, the seller retains the asset if bidders’ synergies are not sufficiently positive. Intuitively, the rents the seller can extract from a bidder are less than its synergy—and the seller optimally extracts only positive rents.

A difference between optimal equity and optimal cash auctions emerges even in this simple setting with ex-ante identical bidders: under mild conditions that restrict the "fatness" of the upper tails of the synergy distribution, I show the threshold synergy value corresponding to the optimal reserve price is lower in optimal equity auctions than in optimal cash auctions. This difference reflects the seller’s ability to extract more rents from equity than from cash bids, which generally lowers the threshold on the synergy value. Hence, with ex-ante identical bidders, optimal equity auctions typically lead to higher social welfare.

When bidders differ ex ante, optimal equity auctions exhibit several additional features. In particular, allocations depend on bidders’ market values: under similar conditions that restrict the "fatness" of the upper tails of synergy distributions, I show bidders with smaller market values are more likely to win, implying optimal equity auctions are allocatively inefficient (the highest-synergy bidder does not always win) when bidders’ market values differ. Intuitively, smaller bidders bid greater equity shares and thus retain smaller equity stakes upon winning, which decreases their informational advantages. Hence the seller can extract larger proportions of rents from smaller bidders, which typically makes it optimal to let smaller bidders win more often. This result contrasts sharply with that in optimal cash auctions, in which

\(^1\)A reserve price in equity auctions corresponds to a "reserve security", possibly bidder-specific, such that the bidder is not allowed to bid a fraction below a particular value.
allocations do not depend on bidders’ market values, because bidders’ informational advantages are independent of their sizes.

For plausible parameterizations, optimal equity and optimal cash auctions can lead to very different allocations. For instance, consider a two-bidder case in which synergies are i.i.d. uniform on $[1,2]$, and the market values of the smaller bidder and the target are both 3. In an optimal cash auction, the allocation is always efficient and each bidder is equally likely to win. However, when the smaller bidder is half the size of the larger bidder in optimal equity auctions, the winning-probability ratio of the smaller bidder over the larger is 1.23, and when the smaller bidder is one quarter the size of the larger, the ratio increases to 1.44. This contrast highlights the extent to which allocations in optimal equity auctions can favor smaller bidders.

An interesting special case of optimal equity auctions obtains in the limit where bidders’ market values far exceed the target’s market value and the extent of the synergies: both the allocation and the seller’s revenue in optimal equity auctions approach those in optimal cash auctions. Intuitively, when bidders’ market values are very large, the fraction of equity the winner pays is tiny, and the winner retains almost all of its synergy, just as in cash auctions. Thus differences in bidders’ informational advantages between equity and cash auctions vanish, and optimal equity auctions degenerate to optimal cash auctions.

In another limiting scenario in which bidders’ market values are far less than the target’s, optimal equity auctions achieve the first-best outcome: the seller can extract all rents. The intuition is that bidders offer almost all of their equities if their market values are small, retaining almost none of their private synergies. Thus, bidders’ rents approach zero, allowing the seller to keep all surplus.

In addition to responding to bidders’ market values, optimal equity auctions also respond to bidders’ synergy distributions. The responses are in the same general directions as those of optimal cash auctions, but the magnitudes differ: given the same market value for bidders, optimal equity auctions typically result in fewer inefficiencies. Intuitively, bidder heterogeneity in synergy distributions spreads bidders’ informational advantages—and inefficient allocations exploit such dispersion. As equity auctions reduce bidders’ informational advantages—and hence the degree of dispersion—optimal equity auctions generally result in more efficient allocations than optimal cash auctions when bidders differ only in synergy distributions.

Note the two forms of bidder heterogeneity have different impacts on the efficiency
of the optimal auction: (sufficient) heterogeneity in bidder size makes optimal equity auctions less efficient than optimal cash auctions, whereas heterogeneity in the distributions of bidder valuations leads to more efficient allocations. This difference reflects that heterogeneity in market values disperses bidders’ informational advantages only in equity but not in cash auctions, which typically makes it attractive to favor smaller bidders (with less informational advantages) in the equity auction. In contrast, heterogeneity in synergy distributions affects bidders’ informational advantages in both cash and equity auctions, and differences in bidders’ informational advantages are reduced in equity auctions relative to cash auctions, precisely because bidders do not retain their full synergies (equity stake) upon winning. Such contrasts highlight how the optimal design of equity auctions hinges sensitively on the nature of bidder heterogeneity.

Optimal equity auctions account for both sources of ex-ante bidder heterogeneity, achieving superior revenues. Consequently, they generate higher expected revenues than any other formats that either do not adjust for bidder heterogeneity or adjust for only a single source of bidder heterogeneity. Furthermore, optimal equity auctions always generate higher expected revenues than optimal cash auctions, regardless of how much bidders differ ex ante. In contrast, any equity-auction format that adjusts for only a single form of bidder heterogeneity can generate lower expected revenues than optimal cash auctions when the other form of bidder heterogeneity is substantial.

In addition to providing these intuitive characterizations of optimal equity auctions, my analysis also reveals features of equity auctions that are less intuitive at first glance. Observe that some of the results require conditions that constrain the upper tails of the synergy distributions; in fact, I show that when these conditions are not met, that is, when the upper tails of the distributions of synergies are sufficiently fat, counter-intuitive results can obtain. For example, consider the seemingly intuitive results that (1) the optimal reserve price corresponds to a lower synergy value in equity than in cash auctions because the seller can extract more rents from equity bids, and (2) the allocation of optimal equity auctions favors smaller bidders because the seller can extract a larger proportion of their rents. Both results can be reversed (so that the probability of trade is less in optimal equity than in optimal cash auctions, and smaller bidders are preferred even when bidders have the same synergy distributions and synergy values) when synergy distributions have fat upper tails. Such counter-intuitive results can arise because of the very fact that equity bids tie payments to the winner’s actual value. This dependence makes a seller’s revenues
more sensitive to the upper tails of synergies in equity than in cash auctions. I will show that when the tails are especially fat, this sensitivity becomes so important that the properties of optimal equity auctions no longer reflect the direct first-order intuition. Such "tail effects" highlight the richness in the implications that the formulation of optimal equity auctions can generate, offering deeper insights into the working of equity auctions at more subtle levels.


These papers examine optimal mechanisms in cash auctions. In contrast, I explore the optimal mechanism when payments are equities. What complicates the analysis is that the approaches for deriving optimal cash mechanisms do not directly apply due to the dependence of equity bids’ values on bidders’ private types. Nonetheless, through a set of transformations on bidders’ incentive conditions and on the seller’s objective function, I show the concept of virtual valuation still holds, and via this concept, the optimal design can again be formulated. The virtual valuation in equity auctions has a finer structure than its cash auction counterpart, and I derive the rich set of implications of this structure.

Hansen (1985) was the first to examine security-bid auctions, demonstrating equity bids generate higher expected revenues than cash bids. Cremer (1987) shows full rent extraction with negative cash transfers. Samuelson (1987) examines how moral hazard and adverse selection may lead to inefficient selections. Riley (1988) shows adding royalties to cash payments increases expected revenues. Rhodes-Kropf

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²Studies have also analyzed revenue consequences in cash auctions in many other situations, for example, preemptive bidding (Fishman 1988, Hirshleifer and Png 1989), sequential auctions (Bernhardt and Scoones 1994), and bidder cross-shareholdings (Dasgupta and Tsui 2004).

DeMarzo, Kremer, and Skrzypacz (2005) study the general class of securities auctions and show that when the seller restricts bids to an ordered set and uses a standard auction format, steeper securities yield higher revenues, and the first-price auction with call options yields the highest revenue over a general set of auction mechanisms. Their paper and mine solve for optimal selling mechanisms under orthogonal constraints: They consider ex-ante identical bidders, comparing different sets of securities and focusing on the problem of security design, whereas I examine equity auctions, considering ex-ante heterogeneous bidders and investigating the associated mechanism-design problem.

My paper is organized as follows. Section 2 describes the model. Section 3 solves for optimal equity auctions. Section 4 derives their properties and implications. Section 5 generalizes the analysis to broader sets of securities than equities. Section 6 concludes. Appendix A further extends the model to incorporate cash payments. Appendix B provides additional details on the analysis.

2 The Model

A group of $n$ risk-neutral bidders bids for a target. The target and bidders have standalone market values $V_T$ and $V_i$ ($i = 1, ..., n$). Bidder $i$ values the target at $V_T + s_i$,

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3Bidding with cash when bidders have limited liability is similar to bidding with debts.

4They also examine informal auctions in which bidders offer arbitrary securities and show such auctions lead to the lowest possible revenues.
where \( s_i \) is its synergy, i.e., the value enhancement the bidder can bring. Bidder \( i \) privately observes \( s_i \), independently drawn from cumulative distribution \( F_i \) (with p.d.f. \( f_i \)) with full support on \( [\underline{s}_i, \overline{s}_i] \), where \( \underline{s}_i \geq 0 \). The values of \( V_T, V_i \), and the functional forms of \( F_i \) (\( i = 1, \ldots, n \)) are common knowledge. The model accommodates ex-ante heterogeneous bidders: \( V_i \) and \( F_i \) can be different for each bidder.

**Definition 1** Bidders are ex-ante homogeneous if and only if \( V_i = V_j \) and \( F_i = F_j \) for all \( i, j \). Bidders are ex-ante heterogeneous if they are not ex-ante homogeneous.

The target is sold via an auction. The sole restriction on the auction format is that bidders (the winner and losers) pay the target with equities, and that a Nash equilibrium exists to the auction design. In an extension (Appendix A), I allow bidders to pay with combinations of cash and equity. I show it is optimal for the seller to accept only equity payments with no cash components.

I pose the model in the context of takeover auctions. This choice is only for expositional convenience. Importantly, my model applies generally to the sale of any non-divisible asset through equity payments, including oil leases auctions, procurements auctions, venture capital financing, lead-plaintiff auctions, project-rights auctions, etc. For example, all results hold in project-rights auctions upon replacing the bidder’s market value with its investment cost, the synergy with the net present value of the project under its control, and the target’s market value with the investment cost the seller incurs (the case where the seller incurs no costs, as in many project-rights auctions, would correspond to a case in which the target’s standalone market value is zero).

### 3 Optimal Mechanisms

Before presenting the technical details of the analysis, I provide examples in section 3.1 to illustrate revenue differences between cash and equity auctions for ex-ante homogeneous and ex-ante heterogeneous bidders, respectively. I derive the general properties for incentive compatible mechanisms in section 3.2 and solve for the optimal mechanism in section 3.3.
3.1 Examples with Homogeneous and Heterogeneous Bidders

**Example 1** *(Ex-ante homogeneous bidders):* Two bidders have synergies $s_1$ and $s_2$ uniformly distributed over $[1, 2]$. They and the target have the same market value $V_1 = V_2 = V_T = 3$.

Because bidders have the same synergy distribution, all standard cash auctions generate the same expected revenue (Myerson 1981) of 4.33. Now consider a second-price equity auction: bidders offer fractions of the merged entity, and the bidder with the highest offer wins and pays a fraction equal to the second-highest bid. Truthful bidding is a dominant strategy: bidder $i$ will bid $p_i = \frac{V_T + s_i}{V_i + V_T + s_i}$. For instance, if $s_1 = 1.6$ and $s_2 = 1.5$, bidder 1 bids $\frac{4.6}{7.6}$ and bidder 2 bids $\frac{4.5}{7.5}$. Thus, bidder 1 wins and pays bidder 2’s bid, corresponding to a monetary value of $\frac{4.5}{7.5} \times 7.6 = 4.56$. Integrating over $s_1$ and $s_2$ yields an expected revenue of 4.53.

This example is based on Hansen (1985) and demonstrates how the second-price equity auction generates higher expected revenue than cash auctions. In fact, because bidders are ex-ante homogeneous in this example, all standard equity auctions generate the same expected revenue (DeMarzo, Kremer, and Skrzypacz 2005).

**Example 2** *(Ex-ante heterogeneous bidders): The same as Example 1 except $V_2 = 6$.*

Bidders have different market values in this example. Cash auctions are unaffected and still generate an expected revenue of 4.33. However, different equity auctions will generate different revenues. In particular, the standard second-price equity auction that works well in Example 1 would generate poor revenue, because when bidders’ market values differ, the smaller bidder may win by offering a large fraction of equity that has only a small monetary value.

One natural format that adjusts for bidders’ market values is the following market-cap-adjusted second-price auction: the equity fraction $p_i$ that bidder $i$ offers is evaluated according to a formula $B_i = p_i (V_i + V_T)$, which is an estimate of the offer’s monetary value excluding synergies. If $i$ and $j$ offer the highest and second-highest $B$-values, $i$ wins and pays an equity fraction of $\frac{B_j}{V_T + V_i}$. Truthful bidding is again a dominant strategy. Under optimal reserve prices (Appendix B), the expected revenue is 4.29, lower than that from cash auctions.
Example 2 shows how equity auctions that naturally adjust for ex-ante bidder heterogeneity may nonetheless generate poor revenues, highlighting the importance for the proper auction design when bidders differ ex-ante. This motivates the following questions: When bidders differ ex-ante, what is the optimal mechanism that generates the highest expected revenue? How does the optimal mechanism respond to different forms of ex-ante bidder heterogeneity? Does it always generate higher revenues than optimally designed cash auctions? I next investigate these questions.

3.2 Properties of Incentive Compatible Mechanisms

For notational ease, define \( f(S) \equiv \prod_{i=1}^{n} f_i(s_i) \) to be the joint density of \( S \equiv (s_1, s_2, ..., s_n) \) and \( f_{-i}(S_{-i}) \equiv \prod_{j \neq i} f_j(s_j) \) to be the joint density of \( S_{-i} \equiv (s_1, ..., s_{i-1}, s_{i+1}, ..., s_n) \). Further, let \( N \equiv \{1, 2, ..., n\} \) denote the set of bidders, let \( \chi_i \equiv [\bar{s}_i, \bar{s}_i] \) denote the set of bidder \( i \)’s synergy, let \( \chi \equiv \times_{j=1}^{n} \chi_j \) denote the product of these sets, and let \( \chi_{-i} \equiv \times_{j \neq i} \chi_j \) denote the product of these sets excluding bidder \( i \).

In light of the revelation principle, consider a direct mechanism \((W, Q)\) in which a truth-telling equilibrium exists and (1) \( W : \chi \rightarrow \mathbb{R}_{\geq 0}^n \) is the winning rule, where \( \Sigma_{i=1}^{n} W_i(S) \leq 1 \) for all \( S \) and \( W_i(S) \) is bidder \( i \)’s winning probability when bidders report synergies \( S \); and (2) \( Q : \chi \times N \rightarrow \mathbb{R}_{\geq 0}^n \) is the equity-retention rule: \( Q_i(S, j) \) is the fraction of equity bidder \( i \) retains (i.e., \( 1 - Q_i \) is the equity fraction bidder \( i \) pays) when bidders report \( S \) and bidder \( j \) wins, and

\[
Q_i(S, j) \in [0, 1] \quad (1)
\]

for all \( i, S, \) and \( j \).

Note the equity-retention rule depends explicitly on the winner’s identity. This dependence allows for mechanisms that probabilistically determine the winner and for mechanisms in which losing bidders pay.

Denote bidder \( i \)’s expected profit when it has \( s_i \) but reports \( z_i \), and all other bidders report truthfully by \( v_i(s_i, z_i) \); then

\[
v_i(s_i, z_i) = \int_{\chi_{-i}} \left[ (V_T + V_i + s_i) Q_i(z_i, S_{-i}, i) - V_i \right] W_i(z_i, S_{-i}) f_{-i}(S_{-i}) dS_{-i} + \sum_{j \neq i} \int_{\chi_{-i}} \left[ V_i Q_j(z_i, S_{-i}, j) - V_i \right] W_j(z_i, S_{-i}) f_{-i}(S_{-i}) dS_{-i}. \quad (2)
\]
To express equation (2) more concisely, denote bidder $i$’s winning probability when it reports $z_i$ and all others report truthfully by $G_i(z_i)$:

$$G_i(z_i) = \int_{\chi_{-i}} W_i(z_i, S_{-i}) f_{-i}(S_{-i}) dS_{-i}. \quad (3)$$

Analogously, denote the expected fraction of the merged entity that bidder $i$ retains upon winning if it reports a value $z_i$ and all others report truthfully by $q_i(z_i)$:

$$q_i(z_i) G_i(z_i) = \int_{\chi_{-i}} Q_i(z_i, S_{-i}, i) W_i(z_i, S_{-i}) f_{-i}(S_{-i}) dS_{-i}. \quad (4)$$

Using equations (3) and (4), bidder $i$’s expected profit, equation (2), becomes

$$v_i(s_i, z_i) = \left[ (V_T + V_i + s_i) q_i(z_i) - V_i \right] G_i(z_i) - \omega_i(z_i), \quad (5)$$

where

$$\omega_i(z_i) = \sum_{j \neq i} \int_{\chi_{-i}} V_i (1 - Q_i(z_i, S_{-i}, j)) W_j(z_i, S_{-i}) f_{-i}(S_{-i}) dS_{-i}. \quad (6)$$

The first term on the right-hand side of equation (5) is bidder $i$’s expected profit in the absence of payments when $i$ loses; the second term $\omega_i(s_i)$ is the reduction in the expected profit from payments upon losing.

Denoting bidder $i$’s equilibrium expected profit by

$$u_i(s_i) \equiv v_i(s_i, s_i) = \left[ (V_T + V_i + s_i) q_i(s_i) - V_i \right] G_i(s_i) - \omega_i(s_i); \quad (7)$$

incentive compatibility conditions yield the following relation (Appendix B provides its derivation):

$$u_i(s_i) = u_i(s_i) + \int_{s_i}^{s_i} q_i(t) G_i(t) dt. \quad (8)$$

Equation (8) contains a key intuition: it shows a bidder’s informational rents scale with the extent to which the bidder retains its synergy upon winning, which is the term $q_i$. This scaling property explains why equity auctions typically generate more revenues than cash auctions. The equity retention is 1 in cash auctions but less than
1 in equity auctions. This difference reduces the differential rents a high-synergy bidder earns over a low-synergy bidder in equity auctions, which, under appropriate boundary conditions, reduces bidders’ overall rents and raises the seller’s revenue.

Next, denote the seller’s expected profit by \( \pi_s \):

\[
\pi_s = \sum_{i=1}^{n} \int_{\tilde{s}_i}^{\bar{s}_i} G_i(s_i) s_i f_i(s_i) \, ds_i - \int_{\tilde{s}_i}^{\bar{s}_i} u_i(s_i) f_i(s_i) \, ds_i,
\]

where the first term is the expected increase in social welfare, the second term is the sum of all bidders’ expected profits, and their difference corresponds to the seller’s expected profit. Rewrite equation (9) as

\[
\pi_s = \sum_{i=1}^{n} \pi_{s,i},
\]

where

\[
\pi_{s,i} = \int_{\tilde{s}_i}^{\bar{s}_i} G_i(s_i) s_i f_i(s_i) \, ds_i - \int_{\tilde{s}_i}^{\bar{s}_i} u_i(s_i) f_i(s_i) \, ds_i
\]

represents bidder \( i \)’s contribution to the seller’s expected profit.

Plugging in equation (8), the second term in equation (11) becomes

\[
\int_{\tilde{s}_i}^{\bar{s}_i} u_i(s_i) f_i(s_i) \, ds_i = - \int_{\tilde{s}_i}^{\bar{s}_i} \left( u_i(\bar{s}_i) + \int_{\tilde{s}_i}^{s_i} q_i(t) G_i(t) \, dt \right) d(1 - F_i(s_i)) = u_i(\bar{s}_i) + \int_{\tilde{s}_i}^{\bar{s}_i} (1 - F_i(s_i)) q_i(s_i) G_i(s_i) \, ds_i,
\]

where integration by parts is used. Thus, equation (11) becomes

\[
\pi_{s,i} = \int_{\tilde{s}_i}^{\bar{s}_i} G_i(s_i) s_i f_i(s_i) \, ds_i - \int_{\tilde{s}_i}^{\bar{s}_i} (1 - F_i(s_i)) q_i(s_i) G_i(s_i) \, ds_i - u_i(\bar{s}_i).
\]

Note the right-hand side of (12) depends on \( q_i(s_i) \). In cash auctions, \( q_i(s_i) \) is always 1, independent of \( s_i \). However, when bidders pay with equities as in this study, \( q_i(s_i) \) is less than 1 and not constant.

Equating the right-hand sides of (7) and (8), one has

\[
[(V_T + V_i + s_i) q_i(s_i) - V_i] G_i(s_i) - \omega_i(s_i) = u_i(\bar{s}_i) + \int_{\tilde{s}_i}^{\bar{s}_i} q_i(t) G_i(t) \, dt,
\]

12
where \( \omega_i (\cdot) \) is given by (6). Equation (13) imposes a constraint on \( q_i (\cdot) \) in the form of an integral equation. One would like to eliminate \( q_i (\cdot) \) from (12) via this constraint; however, \( q_i (\cdot) \) appears in (13) in both the linear and the integral terms, and these terms do not combine. Thus, directly substituting (13) into (12) would not eliminate \( q_i (\cdot) \). Note also a technical subtlety that \( q_i (\cdot), G_i (\cdot), \) or \( \omega_i (\cdot) \) may not be differentiable over the full interior of \([\underline{s}_i, \overline{s}_i]\). Thus one needs to work with (13) in its integral rather than differential form.

The following theorem obtains a set of integral transformations on bidders’ incentive conditions and on the seller’s objective function, which eliminate \( q_i (\cdot) \) and express the optimization program in a form that allows for the explicit solution of the optimal mechanism.

**Theorem 1** *(Revenue decomposition and existence of virtual valuation in equity auctions)*: The seller’s expected revenue in equation (10) is

\[
\pi_s = \pi_{s,a} + \pi_{s,b} + \pi_{s,c},
\]

where

\[
\pi_{s,a} \equiv - \sum_{i=1}^{n} u_i (\underline{s}_i) \tau_i,
\]

\[
\pi_{s,b} \equiv - \sum_{i=1}^{n} \int_{\underline{s}_i}^{\overline{s}_i} (1 - F_i (s_i)) \delta_i (s_i) ds_i,
\]

\[
\pi_{s,c} \equiv \int_{X} \left[ \sum_{i=1}^{n} W_i (S) \phi_i (s_i) \right] f (S) dS,
\]

\[
\delta_i (s_i) \equiv \frac{\omega_i (s_i)}{V_T + V_i + s_i} + \int_{\underline{s}_i}^{s_i} \frac{\omega_i (t)}{(V_T + V_i + t)^2} dt,
\]

\[
\tau_i \equiv \frac{1}{(V_T + V_i + s_i)} \int_{\underline{s}_i}^{\overline{s}_i} (1 - F_i (s_i)) ds_i + 1,
\]

and \( \phi_i (s_i) \) is the virtual valuation defined in equation (20).

**Definition 2** The virtual valuation in equity auctions is

\[
\phi_i (s_i) \equiv s_i - \frac{V_i (1 - F_i (s_i))}{(V_T + V_i + s_i) f_i (s_i)} - \frac{V_i \int_{s_i}^{\overline{s}_i} (1 - F_i (t)) dt}{(V_T + V_i + s_i)^2 f_i (s_i)}.
\]
Theorem 1 shows the seller’s expected revenue decomposes into the sum of three terms. This decomposition disentangles the effects on the expected revenue from bidders’ individual rationality constraints, losing bidders’ payments, and the allocation of the auction, which allows for the explicit solution of the optimal mechanism. Because \( \tau_i > 0 \) and \( u_i (s_i) \geq 0 \) (required by the bidder’s individual rationality constraint), the maximum possible value for the first term \( \pi_{s,a} \) is zero, which obtains if \( u_i (s_i) = 0 \) for all \( i \). So, too, because all terms on the right-hand side of (6) are nonnegative, \( \omega_i (s_i) \geq 0 \) and \( \delta_i (s_i) \geq 0 \) for all \( s_i \). The maximum possible value for the second term \( \pi_{s,b} \) is zero, which obtains if losing bidders do not pay, i.e., the equity retention \( Q_i (s, j) = 1 \) for all \( i, s, \) and \( j \neq i \). Next, consider the term \( \sum_{i=1}^{n} W_i (S) \phi_i (s_i) \) in \( \pi_{s,c} \). The allocation \( W_i (X) \) is, in effect, a weighting function, and hence giving weight only to the maximal \( \phi_i (s_i) \) is optimal, provided that \( \phi_i (s_i) > 0 \). This would maximize the expression at every point \( S \) and thus would maximize the third term \( \pi_{s,c} \).

To better understand Theorem 1, consider the corresponding cash-auctions result for comparison. In deriving optimal cash auctions, Myerson (1981) develops the concept of virtual valuation, showing that it represents the rents the seller can extract from a bidder. Myerson (1981) obtains the form of the virtual valuation as a function of the bidder’s synergy value and synergy distribution:

\[
\psi_i (s_i) \equiv s_i - \frac{1 - F_i (s_i)}{f_i (s_i)} \quad \text{(Myerson 1981).} \tag{21}
\]

When bidders offer securities, on the other hand, the fact that security bids’ values depend on bidders’ private types complicates the analysis of the optimal mechanism, and a priori, it is unclear whether virtual valuations can even be defined in such settings.

Theorem 1 demonstrates an important result that the concept of virtual valuation holds in equity auctions: namely, a function exists for an equity-offering bidder in terms of the bidder’s private type and any common knowledge (e.g., bidders’ synergy distributions, bidders’ and the target’s market values), which measures the rents the seller can extract from the bidder. Equation (20) specifies the form of the virtual valuation, and section 3.3 shows optimal equity auctions can again be formulated via this solution concept. The virtual valuation for equity auctions entails a richer structure than its cash-auction counterpart, with its additional dependence on the market values of the bidder and the target. In section 4, I show this structure leads to a rich set of implications and is crucial for understanding how ex-ante bidder heterogeneity alters...
the optimal auction design. In section 5, I show virtual valuations can also be defined in a broader class of securities than equities, and I generalize Theorem 1 for that class.

3.3 Solutions for Optimal Mechanisms

I now derive optimal mechanisms when the design problem is regular.

Assumption 1 The design problem is regular: \( \phi_i(\cdot) \) increases over \([s_i, \bar{s}_i]\) for all \( i \).

The regularity condition generally holds as long as the distribution \( f_i(\cdot) \) does not decrease too quickly. For example, uniform distributions satisfy the regularity condition. In general, this regularity condition is easier to satisfy than its analogue in cash auctions for the following reason. In both equity and cash auctions, a bidder’s virtual valuation is smaller than its sympathy \( s_i \), and the difference represents the bidder’s rents. Because the sympathy increases in itself at a rate of one, the regularity condition holds unless this difference (between the sympathy and the virtual valuation) decreases in \( s_i \) with a rate of more than one. Because the bidder captures less rent in equity auctions, this difference—and its rate of change—is generally smaller and hence the regularity condition is more likely to hold. In particular, when the bidder is much smaller than the seller, the seller extracts almost full rents from the bidder (Corollary 3) and the regularity condition holds for any distribution \( F_i(\cdot) \).

I assume the regularity condition holds in the rest of the paper, which simplifies the analysis. Under this condition, an individually rational and incentive compatible mechanism exists that maximizes \( \pi_{s,a}, \pi_{s,b}, \) and \( \pi_{s,c} \) simultaneously; it follows that the mechanism maximizes the sum of \( \pi_{s,a}, \pi_{s,b}, \) and \( \pi_{s,c} \) and is therefore optimal.

I first present a necessary condition for the optimal mechanism.

Lemma 1 Losing bidders never pay in the optimal mechanism: The equity-retention rule upon losing is

\[
Q_i(S, j) = 1 \quad \text{for all } j \neq i.
\]  

(22)

Lemma 1 shows losing bidders never pay in optimal equity auctions. The intuition for this result reflects the fact that such payments typically increase the amount of equity a bidder retains upon winning. As informational rents scale with the equity retention, the bidder’s rents increase, thereby reducing the seller’s revenues. This result contrasts with optimal cash auctions, in which losing bidders may also pay (e.g.,
all-pay auctions), because bidders in cash auctions retain all of their equities upon winning—hence their informational advantages are unaffected by whether they would pay upon losing.

I now present the main result of the paper: the formulation of optimal equity auctions.

**Theorem 2** (The set of all optimal mechanisms): A direct mechanism \((W, Q)\) admits a truthful-reporting equilibrium and is optimal if and only if

(i) the winning rule is

\[
W_i(S) = \begin{cases} 
1 & \text{if } \phi_i(s_i) > \max_{j \neq i} \{\phi_j(s_j)\} \text{ and } \phi_i(s_i) \geq 0 \\
0 & \text{if } \phi_i(s_i) < \max_{j \neq i} \{\phi_j(s_j)\} \text{ or } \phi_i(s_i) < 0
\end{cases},
\]  

(23)

where \(\phi_i(s_i)\) is the virtual valuation in equation (20), for all \(i\) and \(S\),

(ii) losing bidders do not pay, or, equation (22) holds, and

(iii) the equity-retention rule upon winning satisfies

\[
\int_{X_{-i}} Q_i(s_i, S_{-i}, i) W_i(s_i, S_{-i}) dS_{-i} = \frac{V_i G_i(s_i)}{V_T + V_i + s_i} + \int_{s_i}^{s_i} \frac{V_i G_i(t)}{(V_T + V_i + t)^2} dt
\]  

(24)

for all \(i\) and \(s_i\), where \(G_i(\cdot)\) is given by (3) and (1) holds for all \(i, S\) and \(j\).

**Corollary 1** (An optimal mechanism): the winning and payment rules in (i) and (ii) of Theorem 2, and the following equity-retention rule upon winning

\[
Q_i(S, i) = \frac{V_i}{V_T + V_i + y_i(S_{-i})},
\]  

(25)

where

\[
y_i(S_{-i}) \equiv \phi_i^{-1} \left( \max \left\{ \max_{j \neq i} \{\phi_j(s_j)\}, 0 \right\} \right)
\]  

(26)

is the minimum value of \(s_i\) that corresponds to a positive virtual valuation and allows \(i\) to win against \(S_{-i}\), and \(\phi_i^{-1}(\cdot)\) denotes a bounded inverse of \(\phi_i(\cdot)\):

\[
\phi_i^{-1}(x) \equiv \begin{cases} 
\bar{s}_i & \text{if } x < \phi_i(s_i) \\
\underline{s}_i & \text{if } x > \phi_i(s_i)
\end{cases},
\]  

(27)

\(y \in [\underline{s}_i, \bar{s}_i] \text{ s.t. } \phi_i(y) = x \text{ if } x \in [\phi_i(\underline{s}_i), \phi_i(\bar{s}_i)]\)

constitute an optimal mechanism.
Part (i) of Theorem 2 shows optimal mechanisms select the bidder with the highest virtual valuation as the winner, provided the highest virtual valuation is positive. This result is intuitive because the virtual valuation represents the rents the seller can extract from a bidder (equation (17)); therefore it maximizes the seller’s revenue to select the bidder with the highest virtual valuation. Section 4 discusses virtual valuation and the properties of optimal mechanisms in details.

Part (ii) of Theorem 2 shows losing bidders never pay in optimal mechanisms, consistent with Lemma 1. Part (iii) of Theorem 2 shows the equity-retention rule upon winning can be implemented in multiple ways, as long as equation (24) holds. Equation (25) in Corollary 1 is one such implementation; the mechanism is analogous to the standard second-price auction in that the winner’s payment depends on the synergy of the bidder with the second-highest virtual valuation, and truth-telling is a weakly dominant strategy. In fact, when bidders are ex-ante homogeneous, this mechanism is precisely the standard second-price auction with an optimal reserve price. Alternatively, optimal mechanisms can be implemented in a way analogous to the standard first-price auction, in which the winner’s payment depends on its own synergy but not on others’ synergies.

4 Properties and Implications

The virtual valuation $\phi_i(s_i)$ in equation (20) is an important component of this paper’s analysis. Whereas its counterpart for cash auctions depends only on the bidder’s synergy value and synergy distribution, the virtual valuation for equity auctions exhibits a key difference in that it depends additionally on the bidder’s and target’s market values. Such dependence provides an integrated structure that links equity and cash auctions, nesting the virtual valuation for cash auctions as a special case while offering new implications for equity auctions.

Below, I first examine the limiting behaviors of the virtual valuation before investigating its general properties.

**Corollary 2** For all bidders $i$, synergies $s_i$ and synergy distributions $F_i(\cdot)$, $\lim_{\tau \to \infty} \phi_i(s_i) = \psi_i(s_i)$, where $\psi_i(s_i)$ is the virtual valuation for cash auctions in equation (21).

Corollary 2 shows when a bidder’s market value is much larger than the target’s market value and the extent of the synergy, its virtual valuation in equity auctions
approaches that in cash auctions. This result reflects that the fraction of equity a large bidder would pay upon winning is close to zero. Therefore, the bidder would retain almost the full extent of its synergy and thus would earn the same rents as in cash auctions.

Corollary 2 adds flexibility to the results. In optimal equity auctions (Theorem 2), any winning bidder pays with equity. However, Corollary 2 shows the limiting situation of $V_i \to \infty$ corresponds to bidder $i$ paying with cash. Thus, upon taking such limits, the formulation of optimal equity auctions applies to settings in which some bidders must offer cash (Proposition 6 in Appendix A explicitly derives optimal mechanisms in such settings). In particular, if all bidders’ market values are large, optimal equity auctions degenerate to optimal cash auctions: both the allocation and the seller’s revenue in optimal equity auctions approach those in optimal cash auctions.

Optimal cash auctions represent the lower bound on the expected revenues that optimal equity auctions can generate. In another limiting scenario, the upper bound on the expected revenues obtains.

**Corollary 3** For all bidders $i$, synergies $s_i > 0$ and synergy distributions $F_i(\cdot)$,

$$\lim_{V_i \to 0} \phi_i(s_i) = s_i.$$

Corollary 3 shows when a bidder is much smaller than the target, its virtual valuation approaches its synergy. This represents the first-best outcome of full rent extraction. Intuitively, a small bidder would offer nearly 100% of equity and thus retain almost none of its private synergy, allowing the seller to extract close to full surplus. This full-extraction result is related to Cremer’s (1987) result of full extraction with sufficiently negative cash transfers, because negative cash transfers effectively reduce bidders’ market values.

**Corollary 4** When the sum of $V_T$ and $V_i$ is much larger than the extent of synergy, the virtual valuation approaches $s_i - \frac{V_i}{V_T+V_i} \frac{1-F_i(s_i)}{f_i(s_i)}$. Or, for all bidders $i$, synergies $s_i > 0$ and synergy distributions $F_i(\cdot)$,

$$\lim_{V_T+V_i \to \infty} (\phi_i(s_i) - s_i + \frac{V_i}{V_T+V_i} \frac{1-F_i(s_i)}{f_i(s_i)}) = 0.$$

To understand Corollary 4, note that the difference between $s_i$ and virtual valuation represents the bidder’s rents. In cash auctions, this difference is $\frac{1-F_i(s_i)}{f_i(s_i)}$. In equity auctions, this difference is generally smaller because the winning bidder retains only a fraction of the equity and hence a fraction of the rents. If $V_T$ and $V_i$ are much larger than the extent of the synergy, the fraction of equity the bidder retains is a
constant \( \frac{V_i}{V_T + V_i} \), independent of the bidder’s and other bidders’ synergies. Because the bidder’s rents are proportional to this fraction, the difference between \( s_i \) and virtual valuation becomes \( \frac{V_i}{V_T + V_i} \frac{1-F_i(s_i)}{f_i(s_i)} \), as the corollary shows.

In the general case when the bidder’s and the target’s market values are finite, \( \phi_i (s_i) \) is given by equation (20). The second term \( \frac{V_i}{V_T + V_i} \frac{1-F_i(s_i)}{f_i(s_i)} \) is similar to the term \( \frac{V_i}{V_T + V_i} \frac{1-F_i(s_i)}{f_i(s_i)} \) in Corollary 4, where the factor \( \frac{V_i}{V_T + V_i} \) is the fraction of equity the bidder would keep if the bidder breaks even upon winning. This factor endows the virtual valuation with a scaling property, making the virtual valuation decrease in the extent the bidder would retain its synergy upon winning. This scaling property leads to the limiting behaviors of the virtual valuation examined earlier, and it has additional implications I will highlight.

The third term in the virtual valuation represents a correction to the fact that the actual fraction of equity the bidder keeps upon winning is less than \( \frac{V_i}{V_T + V_i} \) and is not a constant.\(^5\) Note that this third term depends on the synergy distribution through \( \int_{\bar{s}_i}^{s_i} (1-F_i(t)) \frac{dt}{f_i(s_i)} \), whereas the second term in the virtual valuation for cash (or equity) auctions depends on the synergy distribution through \( \frac{1-F_i(s_i)}{f_i(s_i)} \). Thus, the relative weight of the third to the second term tends to increase in the "fatness" of the upper tails (i.e., \( \bar{s}_i - s_i \)) in the synergy distribution. Intuitively, this feature is due to the underlying difference between equity and cash auctions in that equity bids tie payments to the winner’s actual value, whereas cash bids do not. Thus, the seller’s rents, and hence the virtual valuations, are more sensitive to the fatness of the upper tails in the bidders’ synergy distributions in equity than in cash auctions. Such tail effects manifest in this third term, because this term reflects changes in the seller’s rents due to changes in the equity fraction the bidder offers (i.e., due to the equity fraction not being constant).

When the extent of synergy is small, this third term is small. More specifically, because \( F_i (\cdot) \) is nondecreasing, one has

\[
\int_{s_i}^{\bar{s}_i} (1-F_i(t)) \, dt \leq (1-F_i(s_i)) (\bar{s}_i - s_i),
\]

and the ratio of the third over the second term in the virtual valuation for equity.

\(^5\) This term would not arise were the equity fraction constant: suppose bidder \( i \) bids with a combination of equity and cash, where the equity fraction is \( \alpha_i \) that is fixed for each bidder (i.e., a constant royalty rate). One can show the virtual valuation is then \( \phi_i (s_i) = s_i - (1-\alpha_i) \frac{1-F_i(s_i)}{f_i(s_i)} \). In general, the second term in the virtual valuation represents a "royalty-rate effect", and the third term arises because the actual "rates" are not constant.
(1) If, instead, the extent of synergy is large, the “tail effect” in the third term of the virtual valuation becomes important. I will show that a sufficiently fat upper tail can cause the third term to dominate the second term and lead to counter-intuitive results.

The second term in the virtual valuation for equity auctions (without the minus sign) is always smaller than the second term in the virtual valuation for cash auctions. Further, the third term is usually smaller than the second term unless the synergy distribution has a fat upper tail. This leads to the following sufficient conditions for virtual valuations in equity auctions to exceed their cash-auction counterparts:

**Lemma 2** For any bidder \( i \), synergy \( s_i \), and synergy distribution \( F_i(\cdot) \), the bidder’s virtual valuation for equity auctions exceeds its cash-auction counterpart, i.e., \( \phi_i(s_i) > \psi_i(s_i) \) if either (i) \( s_i - 2s_i \leq V_T \) or (ii) the synergy distribution density \( f_i(\cdot) \) is nondecreasing and \( \psi_i(s_i) \geq 0 \).

Lemma 2 reflects the fact that the seller can extract more rents from equity bids. The lemma has implications for the trading probability in optimal equity auctions. In both optimal equity and optimal cash auctions, if the highest virtual valuation among all bidders is negative, the seller retains the asset even when bidders’ synergies are positive and a trade would result in social gains. Because virtual valuations are higher in equity auctions than in cash auctions under the conditions in lemma 2, socially beneficial trade occurs more frequently in optimal equity auctions under similar conditions:

**Proposition 1** The probability that the asset is sold in optimal equity auctions is at least as high as in optimal cash auctions if for all bidders \( i \), either \( s_i - 2s_i \leq V_T \) or the synergy distribution density \( f_i(\cdot) \) is nondecreasing.

The virtual valuation for equity auctions has another feature that differs from its cash-auction counterpart: it depends on the bidder’s market value. From equation (20), \( \phi_i(s_i) \) depends on \( V_i \) in the second and third terms. The second term (with its minus sign) decreases in \( V_i \); the third term, however, may either decrease or increase in \( V_i \), depending on the value of \( s_i \). Because the third term is typically small, \( \phi_i(s_i) \) tends to decrease in \( V_i \). Formally, Lemma 3 establishes sufficient conditions under which \( \phi_i(s_i) \) decreases in \( V_i \) for all \( s_i \).
**Lemma 3** For any bidder $i$ and $s_i \in (\underline{s}_i, \overline{s}_i)$, \( \frac{\partial \phi_i(s_i)}{\partial V_i} < 0 \) if any of the following conditions hold: (i) $V_T + \underline{s}_i > V_i$, or (ii) $\overline{s}_i - 2\underline{s}_i \leq V_T$, or (iii) the bidder’s synergy is uniformly distributed and $\phi_i(s_i) \geq 0$.

Under similar conditions, if bidders have the same synergy distributions but different market values, smaller bidders are more likely to win in optimal equity auctions.

**Proposition 2** Suppose bidders $i$ and $j$ have the same synergy distributions but different market values with $V_i > V_j$. Then bidder $j$ with any given synergy is more likely to win in optimal equity auctions than bidder $i$ with the same synergy if any of the following conditions hold: (i) $V_T + \underline{s}_i > V_i$, or (ii) $\overline{s}_i - 2\underline{s}_i \leq V_T$, or (iii) their synergies are uniformly distributed.

The intuition reflects that upon winning, a smaller bidder pays a larger equity fraction and hence retains a smaller equity stake. Because bidders’ informational advantages scale with the equity retention, a seller can extract more rents from smaller bidders, making it optimal to let them win more often. This result sharply contrasts with that in optimal cash auctions, in which the allocations do not depend on bidders’ market values because all bidders retain 100% equity stakes, independently of their sizes.

For plausible parameterizations, optimal equity and optimal cash auctions can lead to significantly different allocations. For instance, consider a two-bidder case in which synergies are i.i.d. uniform on [1,2] and the market values of the smaller bidder and the target are both 3. Each bidder is always equally likely to win in optimal cash auctions, but when the smaller bidder is half the size of the larger bidder in optimal equity auctions, the winning-probability ratio of the smaller bidder over the larger is 1.23, and when the smaller bidder is one quarter the size of the larger, the ratio increases to 1.44. Such contrasts highlight the extent to which allocations of optimal equity auctions can favor smaller bidders.

Proposition 2 provides important guidance to the proper design of equity auctions in the presence of heterogeneity in bidders’ market values. It also offers intuition as to why certain (improperly designed) formats perform poorly. For instance, Example 2 demonstrates the market-cap-adjusted second-price auction, which adjusts for bidders’ differing market values in seemingly natural manners, generates poor revenues when bidders’ market values differ substantially. The cause for its poor performance lies in the fact that its allocation favors larger bidders (Appendix B provides details),

21
which goes in the opposite direction as that of optimal equity auctions. Because the
seller can extract fewer rents from larger bidders, letting these bidders win more often
undermines the seller’s revenues.

In addition to responding to bidders’ market values, optimal equity auctions also
respond to bidders’ synergy distributions. Suppose bidders have the same market
values and that their synergies have the same expected values but different disper-
sion: those bidders with less dispersed synergies are favored by the seller and tend to
be more likely to win. More generally, similar to what Myerson (1981) demonstrates
in cash-auction settings, stochastic dominance in the upper tail of the synergy dis-
tribution is relevant: given the same synergies, the allocation tends to favor bidders
whose synergy distributions are conditionally first-order stochastically dominated.
Intuitively, such bidders effectively represent an outside option, allowing the seller
to be more aggressive with other bidders and demand higher bids from them. This
intuition is much the same as that for setting high reserve prices, which allows a seller
to extract more rents despite an efficiency loss.

The response of optimal equity auctions to bidders’ differing synergy distributions
is in the same general direction as that of optimal cash auctions, because virtual val-
uations in both auctions share the same term \[ \frac{1-F_i(s_i)}{f_i(s_i)} \]. However, their magnitudes
differ. Due to the scaling factor \[ \frac{V_i}{V_i + V_i + s_i} \] in the virtual valuation for equity auctions,
the resulting inefficiencies are generally less in optimal equity than in optimal cash
auctions. Intuitively, in equity auctions, the seller can extract more rents from bid-
ders; hence, virtual valuations tend to be closer to bidders’ actual synergies, and
differences in them due to bidders’ differing synergy distributions fall. As the seller
optimally selects the winner based on the virtual valuations, optimal equity auctions
generally lead to more efficient allocations than optimal cash auctions.

Given bidders’ synergy distributions, a natural way to rank the efficiencies of two
mechanisms is to compare the corresponding social welfare. Below, I define a stronger
notion of efficiency ordering: one mechanism is \emph{ex-post more efficient} than another if it leads to higher social welfare at all synergy realizations.

\textbf{Definition 3} Let A and B be two mechanisms in which the winner is selected deter-
mministically, or ties occur with zero probability. Given bidders’ synergy distributions,
mechanism A is \emph{ex-post more efficient} than B if the following holds at all synergy real-
izations \((s_1, s_2, ..., s_n)\): (i) if the asset is sold in B, it is also sold in A, and the winner’s
synergy is no less than the synergy of the winner in B; (ii) if the asset is not sold in
Proposition 3 Suppose bidders have the same market values. (i) Optimal equity auctions are ex-post more efficient than optimal cash auctions if bidders’ synergies are uniformly distributed (their support may differ). (ii) Let $V_T = kV_T^*$ and $V_i = kV_i^*$ ($i = 1, 2, ..., n$), where $k$ is a scaling factor, and $V_T^*$ and $V_i^*$ are constants. Then for any synergy distributions $F_1, F_2, ..., F_n$, optimal equity auctions are ex-post more efficient than optimal cash auctions when $k$ is arbitrarily large. (iii) For any synergy distributions, optimal equity auctions are ex-post more efficient than optimal cash auctions when bidders’ market values are much smaller than the target’s.

Note the two forms of bidder heterogeneity have different impacts on the efficiency of the optimal auction: (sufficient) heterogeneity in bidder size makes optimal equity auctions less efficient than optimal cash auctions, whereas heterogeneity in the distributions of bidder valuations leads to more efficient allocations. To understand this contrast, observe that bidder heterogeneity spreads bidders’ informational advantages—and inefficient allocations exploit such dispersion. As heterogeneity in valuation distributions affect bidders’ informational advantages in both cash and equity auctions, and equity auctions reduce bidders’ informational advantages—and hence the degree of the dispersion, optimal equity auctions lead to more efficient allocations than optimal cash auctions when bidders differ only in synergy distributions. In contrast, a bidder’s market value affects its informational advantage in equity but not in cash auctions, making it attractive to favor smaller bidders (with less informational advantages) in equity auctions. Consequently, optimal equity auctions result in less efficient allocations than optimal cash auctions when bidders (sufficiently) differ in market values. These differences highlight how the optimal design of equity auctions hinges sensitively on the nature of bidder heterogeneity.

Optimal equity auctions account for both sources of ex-ante bidder heterogeneity, maximally exploiting the features of equity bids. Consequently, optimal equity auctions generate higher expected revenues than cash auctions of any format.

Proposition 4 Optimal equity auctions generate strictly higher expected revenues than optimal cash auctions, regardless of any ex-ante heterogeneity in bidders.
In symmetric settings with ex-ante homogeneous bidders, the revenue advantages of equity over cash auctions have been well established in the security-bid auctions literature. Proposition 4 generalizes the revenue advantages of equity auctions to settings with ex-ante heterogeneous bidders. Importantly, the proposition does not follow from an argument that equity bids always generate higher revenues than cash bids; this argument no longer holds when bidders differ ex-ante, as Example 2 highlights (section 7.1 in Appendix A elaborates further on the complications with such arguments). Rather, Proposition 4 reflects the properties of optimally designed equity auctions, which overcome complications from the different sources of bidder heterogeneity.

Proposition 4 shows that optimal equity auctions always generate higher revenues than optimal cash auctions, regardless of how much bidders differ ex-ante. In contrast, alternative equity-auction formats (e.g., a market-cap-adjusted second-price auction) suboptimally adjust for bidder heterogeneity in market values, and do not adjust for heterogeneity in synergy distributions. Consequently, the alternative formats generate expected revenues lower than optimal equity auctions when either source of the heterogeneity is present. Moreover, they can generate expected revenues lower than optimal cash auctions when bidder heterogeneity is substantial.\(^6\)

**Anomalies.** Lemmas 2, 3, Proposition 1 are established under their corresponding sufficient conditions on the “fatness” of the upper tails of synergies relative to the size of the target. One might conjecture that these results follow from the casual intuition that the seller can extract more rents from equity bids than cash bids, and hence that no conditions need be imposed to ensure these results. Here I show such conjectures are false, and that the formulation of optimal equity auctions has deeper implications than just this first-order intuition.

Specifically, as discussed earlier, the relative importance of the third term in the virtual valuation for equity auctions increases in the size of the upper tails of the synergy distribution. With sufficiently high tails, the third term can dominate over the second term and lead to counter-intuitive results, as I show in the example below.

**Example 3 (Counter example):** Suppose bidder 1 has synergy distribution \( f_1 (s_1) = \frac{A}{(s_1 - B)^2} \) over \([\bar{s}, \bar{s}]\), where \(0 < B < \bar{s} < \bar{s}\) and \(A \equiv \frac{(\bar{s} - B)(s - B)}{\bar{s} - \bar{s}}\) normalizes the distribution.

\(^6\)In fact, any equity-auction format that adjusts only for bidders’ market values (and not for bidders’ synergy distributions) can generate revenues lower than optimal cash auctions when bidders’ synergy distributions differ by “enough”. Hence, it is important to account for both sources of bidder heterogeneity in the auction design, as optimal equity auctions do.
Direct calculation yields \(1 - F(s_1) = \frac{A}{s_1 - B} - \frac{A}{s - B}\). Then the virtual valuation for equity and cash auctions are

\[
\phi_1(s_1) = s_1 - \frac{V_1}{V_T + V_1 + s_1} \left( s_1 - B - \frac{(s_1 - B)^2}{s - B} \right) - \frac{V_1 (s_1 - B)^2}{(V_T + V_1 + s_1)^2} \left( \ln \frac{s - B}{s_1 - B} - \frac{s - s_1}{s - B} \right)
\]

and \(\psi_1(s_1) = B - \frac{(s_1 - B)^2}{s - B}\), respectively. Inspection of equation (29) shows that when \(\bar{s}\) (the upper tail) increases arbitrarily, the third term goes to infinity, dominating the second term, which stays finite. This dominance suggests counter-intuitive results may obtain when \(\bar{s}\) is sufficiently high.

In particular, I now show the following can obtain even when \(\phi_1(\cdot)\) and \(\psi_1(\cdot)\) monotonically increase on \([s_l, \bar{s}]\):

1. A virtual valuation in the cash auction can exceed that in the equity auction, i.e., \(\psi_1(s^*) > \phi_1(s^*) > 0\) for some \(s^*\).

2. The probability of trade may be lower in the optimal equity auction than in the optimal cash auction.

3. A bidder’s virtual valuation may increase in its size, i.e., \(\frac{\partial \phi_1(s^*)}{\partial V_1} > 0\) and \(\phi_1(s^*) > 0\) for some \(s^*\). This result implies, given the same synergy distribution and synergy value, larger bidders in optimal equity auctions may be more likely to win.

To establish claims 1 and 2, let \(V_1 = 10, V_T = 1, B = 0.1, \underline{s} = 3, \text{ and } \bar{s} = 100\). Then both \(\phi_1(\cdot)\) and \(\psi_1(\cdot)\) are strictly increasing on \([\underline{s}, \bar{s}]\). At \(s^* = 5, \psi_1(s^*) = 0.34\), which exceeds \(\phi_1(s^*) = 0.15\), demonstrating result 1. Further, suppose bidder 1 is the single bidder (or all other bidders have the same market value and synergy distribution as bidder 1). Because \(\psi_1(3) = 0.18 > 0\), the asset is always sold in optimal cash auctions. However, because \(\phi_1(3) = -0.11 < 0\), the asset is not always sold in optimal equity auctions. Indeed, because \(\phi_1^{-1}(0) = 4.28\), the asset is sold only with probability \(1 - F_1(4.28) = 68\%\) (when bidder 1 is the single bidder), yielding result 2. To establish claim 3, let \(V_1 = 50, V_T = 1, B = 0.1, \underline{s} = 5, \text{ and } \bar{s} = 100\). Then \(\phi_1(\cdot)\) is strictly increasing on \([\underline{s}, \bar{s}]\). At \(s^* = 5, \frac{\partial \phi_1(s^*)}{\partial V_1} = 0.0035 > 0\) and \(\phi_1(s^*) = 0.05 > 0\), demonstrating result 3.

The fat upper tails of the synergy distribution drive these seemingly counter-intuitive results. To understand why optimal equity auctions may result in lower
probabilities of trade than optimal cash auctions, consider a single-bidder setting. In both cash and equity auctions, raising the reserve price has two opposing effects: it increases expected payments conditional on a trade, and it reduces the probability of trade. In equity auctions, a higher reserve price leads to a higher fraction of equity offer. Because the value of equity is tied to a bidder’s actual synergy, the gains from raising the reserve price increases in the extent of the high tails in the synergy distribution. However, in cash auctions, such an increase does not arise, because the value of a cash payment does not depend on a bidder’s synergy. Therefore, with sufficiently fat upper tails, the gains from raising the reserve price become more significant in equity auctions than in cash auctions. This encourages the seller to set a higher reserve price in optimal equity auctions, resulting in a lower probability of trade, contrary to the higher probability of trade that results whenever the upper tail is not too fat (Proposition 1). Similarly, part (iii) of the statement reflects the fact that the high-tail effects in equity auctions become less significant as a bidder’s market values rise.

These counter-intuitive results and the tail effects Example 3 demonstrates offer further insights into the differences between equity and cash bids at a more subtle level, highlighting the richness in the implications the formulation of optimal equity auctions can generate.

5 Generalization Outside of Equities

In this section, I generalize the analysis to a larger class of securities than the class of equities. I show virtual valuations can also be defined for this larger class of securities, and via these valuations, the optimal mechanism can be formulated.

I extend the main model in two ways: I let bidders submit bids from sets of securities formed by linear combinations of two given securities, and I let a bidder’s realized synergy, conditional on the bidder’s private type, be stochastic.⁷ I retain all structure and notation of the main model otherwise.

Let $S(z)$ describe a security, which is the payment to the seller when the merged firm realizes cash flow $z$. For each bidder $i$, let $S_{1i}^*(\cdot)$ and $S_{2i}^*(\cdot)$ be two base securities. Bidder $i$ submits bids from the following set of securities formed by all linear

⁷With equity bids, allowing for stochastic synergies has no impact, because values of equities depend only on the expected value of the synergy, not on the details of the distribution.
combination of the base securities:

\[ S_{ri}(z) = (1 - r) S_{1i}^*(z) + r S_{2i}^*(z) \quad \text{for all } z, \tag{30} \]

where \( r \) indexes the securities. Such a set of securities (in (30)) falls into the convex set of securities introduced in DeMarzo, Kremer, and Skrzypacz (2005), which is equal to its convex hull. Note that \( S_{1i}^*(\cdot) \) and \( S_{2i}^*(\cdot) \) are bidder-specific, i.e., different bidders can submit bids from different sets of securities.

For each bidder \( i \) with private type \( s_i \), let random variable \( Z_{si} \) denote the actual synergy it can realize. Without loss of generality, define the expected value of \( Z_{si} \) to be \( s_i \), i.e., \( s_i = \mathbb{E}[Z_{si}] \) for all \( s_i \in [s_i, \bar{s}_i] \). For all bidders \( i \) and base securities \( j \in \{1, 2\} \), define

\[ K^*_{ji}(s_i) \equiv \mathbb{E} [ S_{ji}^*(V_T + V_i + Z_{si}) | s_i ] , \tag{31} \]

which is the expected payment to the seller when bidder \( i \), whose private type is \( s_i \), pays security \( S_{ji}^* \). Similarly, for all \( i \) and \( r \), define

\[ K_{ri}(s_i) \equiv \mathbb{E} [ S_{ri} (V_T + V_i + Z_{si}) | s_i ] , \]

which is the expected payment to the seller when bidder \( i \) with private type of \( s_i \) pays security \( S_{ri} \). From equations (30) and (31), one has

\[ K_{ri}(s_i) = (1 - r) K_{1i}^*(s_i) + r K_{2i}^*(s_i) = K_{1i}^*(s_i) + r D_{i}^*(s_i) , \]

for all \( r, i, s_i \), where \( D_{i}^*(s_i) \equiv K_{2i}^*(s_i) - K_{1i}^*(s_i) \).

I require \( K_{1i}^*(s_i) \) and \( K_{2i}^*(s_i) \) to be continuous and differentiable, and \( D_{i}^*(s_i) > 0 \) for all \( i \) and \( s_i \). These requirements correspond to mild restrictions on \( S_{1i}^*(\cdot) \), \( S_{2i}^*(\cdot) \), and the distributions of \( Z_{si} \)\(^8\). I assume for simplicity that losing bidders do not pay. This assumption is without loss of generality—as in the earlier analysis, one can show that losers do not pay in the optimal mechanism in security-bid auctions under

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\(^8\)Additional restrictions on \( S_{1i}^*(\cdot) \), \( S_{2i}^*(\cdot) \), and \( r \) are necessary to ensure the feasibility of the securities (for details on the feasibility of securities, see DeMarzo, Kremer, and Skrzypacz 2005). For generality of the formulation of the virtual valuation, I do not impose such restrictions.
Analysis. Consider a direct mechanism \((W, R)\) in which a truth-telling equilibrium exists and (1) \(W: \mathbf{x} \rightarrow \mathbb{R}^n_{\geq 0}\) is the winning rule, where \(\Sigma_{i=1}^n W_i(S) \leq 1\) for all \(S\) and \(W_i(S)\) is bidder \(i\)'s winning probability when bidders report synergies \(S\); and (2) \(R: \mathbf{x} \rightarrow \mathbb{R}^n_{\geq 0}\) is the payment rule: \(R_i(S)\) is the index (i.e., the \(r\)) of the security bidder \(i\) pays upon winning when bidders report \(S\).

Denote bidder \(i\)'s expected profit when it has \(s_i\) but reports \(z_i\), and all other bidders report truthfully by \(v_i (s_i, z_i)\); then

\[
v_i (s_i, z_i) = \int_{\chi-i} [V_T + s_i - (K^*_{1i} (s_i) + R_i (z_i, S_{-i}) D^*_{1i} (s_i)))] W_i (z_i, S_{-i}) f_{-i} (S_{-i}) dS_{-i}.
\]

(32)

To express (32) more concisely, denote the expected value of \(R_i\) that bidder \(i\) pays upon winning if it reports a value \(z_i\) and all others report truthfully by \(r_i (z_i)\):

\[
r_i (z_i) G_i (z_i) = \int_{\chi-i} R_i (z_i, S_{-i}) W_i (z_i, S_{-i}) f_{-i} (S_{-i}) dS_{-i}.
\]

(33)

Using equations (3) and (33), bidder \(i\)'s expected profit is

\[
v_i (s_i, z_i) = [V_T + s_i - (K^*_{1i} (s_i) + r_i (z_i) D^*_{1i} (s_i))] G_i (z_i),
\]

(34)

Denoting bidder \(i\)'s equilibrium expected profit by

\[
 u_i (s_i) \equiv v_i (s_i, s_i) = [V_T + s_i - (K^*_{1i} (s_i) + r_i (s_i) D^*_{1i} (s_i))] G_i (s_i); \]

(35)

incentive compatibility conditions yield the following relation:

\[
u_i (s_i) = u_i (s_i) + \int_{s_i}^{s_i} (1 - K^*_{1i} (t) - r_i (t) D^*_{1i} (t)) G_i (t) dt.
\]

(36)

Using similar techniques as in the preceding analysis, I show the seller’s revenue decomposes and virtual valuations exist in ways analogous to equity auctions:

**Proposition 5** (Generalized virtual valuation): The seller’s expected revenue is

\[
\pi_s = \pi_{s,a} + \pi_{s,c},
\]

(37)

28
where
\[
\pi_{s,a} \equiv - \sum_{i=1}^{n} u_i (\mathbb{E}_i) \left( \frac{1}{D_i^* (\mathbb{E}_i)} \int_{\mathbb{E}_i} (1 - F_i (s_i)) D_i^{*'} (s_i) \, ds_i + 1 \right),
\]
\[
\pi_{s,c} \equiv \int_{\mathcal{X}} \left[ \sum_{i=1}^{n} W_i (S) \hat{\phi}_i (s_i) \right] f (S) \, dS,
\]
and \( \hat{\phi}_i (s_i) \) is the virtual valuation in equation (40).

**Definition 4** The virtual valuation for the set of securities in (30) is
\[
\hat{\phi}_i (s_i) = s_i - \frac{1 - F_i (s_i)} {f_i (s_i)} \left( 1 - K_i^{*'} (s_i) - D_i^{*'} (s_i) \frac{V_T + s_i - K_i^{*} (s_i)} {D_i^* (s_i)} \right) - \\
\frac{\int_{\mathbb{E}_i} (1 - F_i (t)) D_i^{*'} (t) \, dt} {f_i (s_i)} \left[ 1 - K_i^{*'} (s_i) - D_i^{*'} (s_i) \frac{V_T + s_i - K_i^{*} (s_i)} {D_i^* (s_i)} \right] - \\
\frac{D_i^{*'} (s_i)} {D_i^{*2} (s_i)} [V_T + s_i - K_i^{*} (s_i)]
\]

Proposition 5 shows that virtual valuation can be defined for the class of securities in (30), which is larger than the class of equities. As in equity auctions, the optimal mechanism can be constructed via the virtual valuation, analogous to Theorem 2 and Corollary 1. Because the class of securities in (30) can differ for each bidder, the optimal mechanism has a salient feature that accommodates situations in which different bidders offer different types of securities.

The virtual valuation in equation (40) depends on the form of \( K_i^{*} \) and \( D_i^{*} \), which depends on the base securities \( S_{1i}^{*} \) and \( S_{2i}^{*} \) and the conditional distribution of synergy \( Z_{s_i} \). Note that the virtual valuation is a property for the set of the securities. As the set in (30) is spanned by linear combinations of the base securities \( S_{1i}^{*} \) and \( S_{2i}^{*} \), the set is invariant if the base securities themselves undergo similar linear transformations. Indeed, one can verify that the virtual valuation is invariant to such transformations.\(^9\)

The virtual valuation in equation (40) entails a rich structure. Although a detailed examination is outside the scope of this paper, note that equation (40) encompasses the virtual valuations for equity and for cash auctions as special cases. To see this, let \( S_{1i}^{*} \) be the null-security, which pays off zero for any cash flow realizations. Then \( K_i^{*} (s_i) = 0 \) for all \( s_i \). If \( S_{2i}^{*} \) is the full equity, i.e., \( S_{2i}^{*} (z) = z \), then \( D_i^{*} (s_i) = K_i^{*} (s_i) = 0 \). Note also that equation (40) can be expressed in an alternative form as \( \hat{\phi}_i (s_i) = s_i - \frac{\int_{\mathbb{E}_i} D_i^{*'} (t) f_i (t) \, dt} {f_i (s_i)} \left[ 1 - K_i^{*} (s_i) - D_i^{*'} (s_i) \frac{V_T + s_i - K_i^{*} (s_i)} {D_i^* (s_i)} \right] - \\
\frac{D_i^{*'} (s_i)} {D_i^{*2} (s_i)} [V_T + s_i - K_i^{*} (s_i)] \).

\(^9\)
\( V_T + V_i + s_i \). It is straightforward to show equation (40) reduces to (20), the virtual valuation for equity auctions. If, instead, \( S_{2i}^2 (z) = 1 + \alpha (z - V_T - V_i) \), where \( \alpha > 0 \), then \( D_i^s (s_i) = K_{2i}^s (s_i) = 1 + \alpha s_i \). One can show in the limit when \( \alpha \) goes to zero, equation (40) reduces to (21), the virtual valuation for cash auctions.

6 Conclusions

Ex-ante bidder heterogeneity—differences in bidders’ observable characteristics such as size or distribution of valuations—complicates the ranking of security bids and directly impacts the auction revenues. I analyze this impact in equity security-bid auctions.

Given bidder heterogeneity, I derive optimal equity auctions that maximize the seller’s expected revenue. I show how bidder heterogeneity significantly alters the optimal auction design, and how different sources of the heterogeneity have distinctly different implications. In contrast to optimal cash auctions, optimal equity auctions have a key feature that they are allocatively inefficient when bidders’ sizes differ, with the allocation favoring bidders with smaller market values or investment costs. This bias arises because in equity auctions, smaller bidders would retain a smaller equity fraction upon winning, and the equity stake a bidder retains upon winning is tied to the bidder’s informational advantage—the larger the stake, the higher the informational advantage, and the more the bidder’s rents. Thus the seller can extract a larger proportion of rents from smaller bidders, making it optimal to let these bidders win more often. For plausible parameterizations, optimal equity auctions can result in radically different allocations from optimal cash auctions.

If bidders have the same market value but different synergy distributions, the allocations of optimal equity auctions are also inefficient—but less so than in optimal cash auctions. Intuitively, heterogeneity in synergy distributions spreads bidders’ informational advantages in both cash and equity auctions—and inefficient allocations exploit such dispersion. Because equity auctions reduce bidders’ informational advantages—and hence the degree of dispersion among them, optimal equity auctions are more efficient than optimal cash auctions when bidders differ only in synergy distributions.

Optimal equity auctions also have the properties that losing bidders never pay and the reserve prices are typically lower than in optimal cash auctions. Together, these properties allow optimal equity auctions to overcome both sources of bidder hetero-
geneity and to maximally exploit the features of equities. Consequently, when either source of bidder heterogeneity is present, optimal equity auctions generate higher expected revenues than other equity auction formats that suboptimally adjust for bidder heterogeneity. Furthermore, they always generate higher expected revenues than optimal cash auctions, regardless of how much bidders differ ex-ante.

Optimal equity auctions provide a unified framework that links cash and equity auctions under the same structure. When bidders’ market values are much larger than the synergies and the seller’s market value, optimal equity auctions degenerate to optimal cash auctions, achieving the same expected revenue and allocations. If, instead, bidders’ market values are small, optimal equity auctions allow the seller to extract close to full rents, achieving the first-best outcome. This framework offers insights for the design of equity auctions, which are also useful for the design of other types of security-bid auctions.

References


7 Appendix A: Further Extensions

In the main model, bidders pay with equity only. In this section, I extend the model by incorporating cash payments in two ways. Section 7.1 considers optional cash payments and section 7.2 considers mandatory cash payments.

7.1 Optimal Mechanisms When Bidders May Optionally Offer Cash

In this part I assume all bidders (winner and losers) can pay with any combination of cash and equity, where the cash payment is nonnegative. I show it is not optimal for the seller to accept cash payments; thus the optimal mechanism is the same as in the main model.

The analysis follows the same general procedures as in the main model. In the main model, the direct mechanism is represented by \((W, Q)\), here I augment this representation to \((W, Q, M)\), where \(M : \chi \times N \to \mathbb{R}_{\geq 0}^n\) is the cash-payment rule: \(M_i(S, j)\) is the cash amount bidder \(i\) pays when bidders report \(S\) and bidder \(j\) wins. Then equation (2) becomes

\[
\omega_i (z, s) = \int \left[ (V_T + V_i + s_i - \sum M_i(z, S_{-i}, i)) Q_i(z, S_{-i}, i) - V_i \right] W_i(z, S_{-i}) f_{-i}(S_{-i}) dS_{-i} + \sum_j \int [V_i - M_i(z, S_{-i}, j)) Q_i(z, S_{-i}, j) - V_i] W_j(z, S_{-i}) f_{-i}(S_{-i}) dS_{-i} \tag{41}
\]

Then, equation (6) becomes

\[
\omega_i (z) \equiv \int \left[ \sum_j M_i(z, S_{-i}, j) Q_i(z, S_{-i}, j) W_i(z, S_{-i}) f_{-i}(S_{-i}) dS_{-i} + \sum_j \int [V_i (1 - Q_i(z, S_{-i}, j)) + M_i(z, S_{-i}, j) Q_i(z, S_{-i}, j)] W_j(z, S_{-i}) f_{-i}(S_{-i}) dS_{-i} \right] \tag{42}
\]

Following the same procedures as in the main analysis, one can show Theorem 1 holds with equation (42) replacing \(\omega_i\) in equation (18). I show the following:

**Lemma 4** In the optimal mechanism, bidders pay with equity only with no cash component. This result holds independently of any ex-ante bidder heterogeneity.

The no-cash result in Lemma 4 can be understood on multiple levels. On an intu-
itive level, the result reflects the revenue advantages of equity over cash bids (Hansen (1985); Cremer (1987); Riley (1988); DeMarzo, Kremer, and Skrzypacz (2005); among others). More specifically, a bidder’s cash payment would result in the bidder demanding a higher equity share in compensation upon winning, thereby increasing the bidders’ rents (through their increased informational advantages) and reducing the seller’s revenue.

On a technical level, the no-cash result is a consequence of the fact that Theorem 1 holds with equation (42) replacing \( \omega_i \) in equation (18). The validity of Theorem 1 under such replacement shows that the effects of cash payments on expected revenue are separable from other sources. Combined with the fact that an incentive compatible mechanism exists that simultaneously maximizes the contributions to the expected revenue from each source (and that the mechanism features no cash payments), and that positive cash transfers strictly lower the part of contributions from cash payments, it follows that any mechanism with positive cash payments is suboptimal.

Note that I do not allow the seller to pay bidders cash. Without this constraint, negative cash transfers would effectively reduce bidders’ market values, which increases the seller’s revenue; and if the transfers were to exceed bidders’ market values, the seller could extract full rents, as Cremer (1987) has shown. This non-negativity constraint is consistent with the practice: negative cash transfers have not been observed. In fact, even if the seller does offer bidders cash, to extract full rents, the cash offer must exceed the market value of the largest bidder—and bidders in takeovers tend to be larger than the target. In addition, DeMarzo, Kremer, and Skrzypacz (2005) show how moral hazard considerations can preclude negative cash payments.

7.2 Optimal Mechanisms When Some Bidders Must Offer Cash

In this part, I extend the main model in another direction and derive the optimal mechanism in settings in which some bidders must offer cash. Specifically, let \( l \in \{1, 2, ..., n - 1\} \); assume bidders \( i \leq l \) must offer cash, whereas bidders \( i > l \) offer equities. Assume \( \psi_i(\cdot) \) is increasing for all \( i \leq l \) and \( \phi_i(\cdot) \) is increasing for all \( i > l \).

**Proposition 6** Mechanism \((W, M, Q)\) is optimal if and only if
(i) For all $i$, the winning rule is

$$W_i(S) = \begin{cases} 
1 & \text{if } \kappa_i(s_i) > \max_{j \neq i} \{\kappa_j(s_j)\} \text{ and } \kappa_i(s_i) \geq 0 \\
0 & \text{if } \kappa_i(s_i) < \max_{j \neq i} \{\kappa_j(s_j)\} \text{ or } \kappa_i(s_i) < 0 
\end{cases},$$

(43)

where, for all $j$ and $s_j$, $\kappa_j(s_j) \equiv \psi_j(s_j)$ if $j \leq l$ and $\kappa_j(s_j) \equiv \phi_j(s_j)$ if $j > l$.

(ii) For all $i \leq l$ and $s_i$, the cash-payment rule satisfies

$$\int \sum_{j=1}^n M_i(s_i,S_{-i},j) W_j(s_i,S_{-i}) f_{-i}(S_{-i}) dS_{-i} = (V_i + s_i) G_i(s_i) - \int_{S_i}^G G_i(t) dt.$$  

(44)

(iii) For all bidders $i > l$, the bidder does not pay upon losing, and the equity-retention rule upon winning satisfies part (iii) of Theorem 2.

Note that the cash-payment rule in equation (44) corresponds to the equity-retention rule in equation (24) under the limit $V_i \to \infty$. To see this correspondence, suppose $i$ offers equity. Then the cash value of the equity it pays upon winning is $(1 - Q_i(s_i,S_{-i},i))(V_T + V_i + s_i)$. From equation (24), the expected cash value of its equity payment conditional on $s_i$ is

$$(V_T + V_i + s_i) \int (1 - Q_i(s_i,S_{-i},i)) W_i(s_i,S_{-i}) f_{-i}(S_{-i}) dS_{-i}$$

$$= (V_T + V_i + s_i) \left(G_i(s_i) - \frac{V_i G_i(s_i)}{V_T + V_i + s_i} + \int_{S_i}^{s_i} \frac{V_i G_i(t)}{(V_T + V_i + t)^2} dt \right).$$

Upon taking the limit $V_i \to \infty$, the above becomes $(V_i + s_i) G_i(s_i) - \int_{S_i}^{s_i} G_i(t) dt$, which is precisely the right-hand side of equation (44).

8 Appendix B: Alternative Interpretation and Alternative Format

In this appendix, I examine an alternative interpretation of the virtual valuation, discuss an alternative equity-auction format, and provide more details on equation (8).

**Alternative interpretation.** Given the result that virtual valuation for equity auctions exists, as Theorem 1 establishes, the virtual valuation can be alternatively...
interpreted as the marginal revenue in a monopolist pricing situation, the same as what Bulow and Roberts (1989) have shown in cash auctions.

Specifically, consider bidder $i$ in isolation. Suppose the seller makes a take-it-or-leave-it offer that makes the bidder indifferent between accepting and rejecting if it has a synergy $s_i$; then, the seller demands an equity fraction of $\frac{V_T + s_i}{V_T + V_i + s_i}$. The probability the bidder accepts the offer is $p(s_i) \equiv 1 - F_i(s_i)$, which can be interpreted as the quantity demanded by the bidder. Thus, the seller’s revenue is

$$\Pi = (1 - F_i(s_i)) \left( \frac{V_T + s_i}{V_T + V_i + s_i} (V_T + V_i + E[s|s \geq s_i]) - V_T \right)$$

$$= (1 - F_i(s_i)) \left( s_i + \frac{V_T + s_i}{V_T + V_i + s_i} E[s - s_i|s \geq s_i] \right).$$

Taking derivative with respect to $s_i$ yields

$$\frac{d\Pi}{ds_i} = -f_i(s_i) s_i + \frac{V_i (1 - F_i(s_i))}{V_T + V_i + s_i} \left[ \frac{d}{ds_i} \left( \frac{V_T + s_i}{V_T + V_i + s_i} \right) \right] (1 - F_i(s_i)) E[s - s_i|s \geq s_i].$$

(45)

Note $(1 - F_i(s_i)) E[s - s_i|s \geq s_i] = \int_{s_i}^{s_i} (s - s_i) f_i(s) \, ds = -\int_{s_i}^{s_i} (s - s_i) \, d(1 - F_i(s)) = \int_{s_i}^{s_i} (1 - F_i(s)) \, ds$. Further, $\frac{dp}{ds_i} = -f_i(s_i)$ or $\frac{ds_i}{dp} = -\frac{1}{f_i(s_i)}$. The marginal revenue is $\frac{d\Pi}{dp} = \frac{d\Pi}{ds_i} \frac{ds_i}{dp}$, which, upon straightforward algebra, takes the same form as the virtual valuation.

Note the third term in the virtual valuation can be expressed as

$$\frac{ds_i}{dp} \left[ \frac{d}{ds_i} \left( \frac{V_T + s_i}{V_T + V_i + s_i} \right) \right] (1 - F_i(s_i)) E[s - s_i|s \geq s_i].$$

This expression reveals that (1) the third term relates to the rate of change in the equity fraction the bidder offers, as the term $\frac{d}{ds_i} \left( \frac{V_T + s_i}{V_T + V_i + s_i} \right)$ shows, and (2) the third term manifests the tail effects of the synergy distribution, as the term $E[s - s_i|s \geq s_i]$ shows.

**Alternative format.** The following format of equity auction is a variation of the standard second-price auction that adjusts for bidders’ differing market values in natural ways. I impose discriminatory reserve prices to enhance the seller’s expected revenue.

- **Market-cap-adjusted second-price auction.** Each bidder $i$ offers a fraction $p_i$ of equity, but the $p$-offers are evaluated according to a function $B$, where $B(p_i) = p_i (V_i + V_T)$, which is the offer’s monetary value, ignoring synergies. Let $B(p_i)$ be the highest $B$-offer and $\{r_i\}_{i=1}^n$ be the reserve prices. If $B(p_i) \geq r_i$,
i wins and pays an equity fraction $\frac{B^*}{V_T + V_i}$, where

$$B^* = \max \{ r_i, \max_{j \neq i} \{ B(p_j) \} \}.$$  If $B(p_i) < r_i$, no bidder wins. Under this evaluation rule, the bidding strategy is truthful:

$$p_i(s_i) = \frac{V_T + s_i}{V_T + V_i + s_i}.$$  (46)

The equilibrium is typically not efficient: given bidders’ truthful bidding strategies, one has

$$B(p_i) = \frac{V_T + s_i}{V_T + V_i + s_i} (V_T + V_i) = \frac{V_T + s_i}{1 + s_i V_i V_T},$$  which increases in $V_i$. Thus, given the same synergy, larger bidders are more likely to win.

The following lemma derives the optimal reserve prices that maximize the seller’s expected revenue.

**Lemma 5** The following reserve prices are optimal in maximizing the seller’s expected revenue in the market-cap-adjusted second-price auction:

$$r_i = \frac{V_T + \phi_i^{-1}(0)}{V_T + V_i + \phi_i^{-1}(0)} (V_T + V_i).$$

**Derivation of Equation (8).** Incentive compatibility yields

$$u_i(s_i) = \max_{z_i} v_i(s_i, z_i).$$  (47)

By equations (47) and (5), the following holds for all $s_i$ and $s'_i$:

$$u_i(s_i) - u_i(s'_i) = [u_i(s_i) - v_i(s'_i, s_i)] + [v_i(s'_i, s_i) - u_i(s'_i)]$$

$$\leq u_i(s_i) - v_i(s'_i, s_i)$$

$$= (s_i - s_i') q_i(s_i) G_i(s_i).$$

Thus

$$u_i(s'_i) \geq u_i(s_i) + (s'_i - s_i) q_i(s_i) G_i(s_i),$$  (48)

which shows that at all $s_i$, a line at $s_i$ with a slope of $q_i(s_i) G_i(s_i)$ supports the function $u_i(\cdot)$. Thus $u_i(\cdot)$ is convex. Because a convex function is absolutely continuous, it is differentiable almost everywhere in the interior of its domain. By equation (48), at every point that $u_i(\cdot)$ is differentiable, $\frac{du_i(s_i)}{ds_i} = q_i(s_i) G_i(s_i)$. Because every absolutely continuous function is the definite integral of its derivative, equation (8) holds.